Symétries nonrelativistes et gravité de Newton-Cartan

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Unwin, cansado, lo detuvo.
-No multipliques los misterios, le dijo ; estos deben ser simples, recuerda la carta robada de Poe, recuerda el cuarto cerrado de Zangwill.
-O complejos, replicó Dunraven, recuerda el universo.

Agacé, Unwin l'arrêta.
-Ne multiplie pas les mystères, dit-il ; ils doivent être simples, rappelle-toi la carte volée de Poe, rappelle-toi la chambre close de Zangwill.
-Ou complexes, répliqua Dunraven, rappelle-toi l'univers.

Jorge Luis Borges
Abenjacán el Bojarí, muerto en su laberinto, El Aleph (1949)

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Résumé

Bien qu'ayant vu le jour dans un cadre dit relativiste avec l'avènement de la théorie de la relativité générale, le lien intime existant entre géométrie de l'espace-temps d'une part, et gravitation d'autre part, peut se voir étendu aux théories dites nonrelativistes, l'exemple paradigmatique en étant la reformulation géométrique de la gravitation Newtonienne initiée par E. Cartan. De tels espace-temps nonrelativistes diffèrent structurellement de leurs homologues relativistes, ces disparités étant le plus naturellement expliquées en réinterprétant ces premiers comme réduction dimensionnelle d'espace-temps relativistes privilégiés.

L'ambition de cette thèse est double :

Dans une première partie, nous nous intéressons à une généralisation de la classe d'espace-temps relativistes permettant le formalisme ambiant, étudions leur interprétation géométrique ainsi que la classe élargie de structures nonrelativistes pouvant y être plongées.

La seconde partie de ce manuscrit concerne le point de vue, informé par la théorie des groupes, que porte E. Cartan sur la géométrie différentielle et plus précisément l'éclairage que projettent les géométries de Cartan sur les structures nonrelativistes, à la fois dans leur définition intrinsèque et dans leur relation avec des structures relativistes au travers du formalisme ambiant.

Mots clés : Symétries Nonrelativistes, Eisenhart Lift, Gravitation de Newton-Cartan, Réduction Dimensionnelle, Formalisme Ambiant, Géométrie de Cartan.

Abstract

With the advent of general relativity, the profound interaction between the geometry of spacetime and gravitational phenomena became a truism of modern physics. However, the intimate relationship between spacetime geometry and gravitation is by no means restricted to relativistic physics but can in fact be successfully applied to nonrelativistic physics, the paradigmatic example being E. Cartan geometrisation of Newtonian gravity. This geometrisation of nonrelativistic gravitation involves some nonrelativistic structures whose discrepancies in comparison with their relativistic peers are better understood when embedded inside specific classes of relativistic gravitational waves.

The ambition of this Doctoral Thesis is twofold:

In a first part, we discuss a generalisation of the class of gravitational waves allowing the embedding of nonrelativistic features, explore their geometric properties and the new nonrelativistic structures emerging from this study.

In a second part, we advocate how the group-theoretically oriented approach of Cartan to differential geometry can shed new light on nonrelativistic structures, both in an intrinsic and ambient fashion.

Keywords : Nonrelativistic Symmetries, Eisenhart Lift, Newton-Cartan Gravity, Dimensional Reduction, Ambient Formalism, Cartan Geometry.

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Introduction

$\alpha\epsilon\iota$ $\theta\epsilon o\varsigma$ $\gamma\epsilon\omega\mu\epsilon\tau\rho\epsilon\iota$
Toujours le Dieu géométrise.

– Platon, cité par Plutarque dans ses "Quaestiones Convivales" VIII.2. *Moralia*

Physique et Géométrie

Ce travail de thèse traite de Géométrie et de Physique. Nous commencerons donc notre exposé par une brève histoire des liens existant entre ces deux disciplines, avec un intérêt particulier porté à la géométrisation de la notion d'espace-temps. La briéveté de cette introduction ira de pair avec une certaine partialité dont nous ferons délibérément usage afin d'introduire certains des protagonistes récurrents de ce travail (parmi lesquels Platon, Galilée, Newton, Leibniz, Klein et Cartan).

En 300 AEC paraissent les *Éléments*, ouvrage fondateur de la géométrie occidentale, dont la paternité est attribuée à Euclide d'Alexandrie, un étudiant de l'École Platonicienne [1]. Le caractère révolutionnaire de ce traité tient notamment en ce qu'il structure les connaissances mathématiques de l'époque en une présentation axiomatique et logico-déductive, formalisant ainsi un mode de raisonnement au rayonnement désormais universel.

Le rôle central joué par la science géométrique comme instrument de connaissance de la Nature, au sein de la culture grecque ancienne, est capturé par le célèbre aphorisme Platonicien qui sert d'exergue à ce Chapitre. L'idée d'une structure géométrique sous-tendant l'Univers est héritée de la mystique Pythagoricienne et irrigue un pan important de la pensée scientifique occidentale, jusqu'à Galilée, dont la célèbre citation qui suit, issue du traité *L'Essayeur* (1623), enrichit la métaphore médiévale de la Nature comme un livre d'un souffle géométrique :

1. En regard de l'absence de matériaux de première main, la biographie d'Euclide reste sujette à controverse. Cependant, sa participation aux cours de l'Académie (probablement en tant qu'étudiant d'un des étudiants mathématiciens de Platon : Eudoxe de Cnide, Théétète d'Athènes ou Philippe d'Oponte) est bien établie.

> *La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola ; senza questi è un aggirarsi vanamente per un oscuro laberinto.*
>
> *La philosophie est écrite dans ce livre gigantesque qui est continuellement ouvert à nos yeux (je parle de l'Univers), mais on ne peut le comprendre si d'abord on n'apprend pas à comprendre la langue et à connaître les caractères dans lesquels il est écrit. Il est écrit en langage mathématique, et les caractères sont des triangles, des cercles, et d'autres figures géométriques, sans lesquelles il est impossible d'y comprendre un mot. Dépourvu de ces moyens, on erre vainement dans un labyrinthe obscur.*
>
> – Galileo Galilei, *Il Saggiatore* (1623)

On peut citer, parmi les autres figures de cette filiation intellectuelle, l'astronome Johannes Kepler qui, dans son ouvrage *Mysterium Cosmographicum* (1596), proposa un modèle du système planétaire s'appuyant sur les cinq solides Platoniciens.

En 1687 paraissent les trois volumes composant les *Philosophiæ Naturalis Principia Mathematica* dans lesquels Sir Isaac Newton réalise l'ambition programmatique de Galilée de constituer une description *more geometrico* des phénomènes naturels. Ce faisant, Newton ajoute le calcul différentiel et intégral à la liste des "caractères" grâce auxquels s'écrit le grand livre de la Nature [2]. L'application du calcul différentiel à la géométrie, initiée par Newton, fondera les bases de la géométrie différentielle.

Newton propose aux fondements de sa théorie de la mécanique les concepts de temps et d'espace absolus. D'un point de vue géométrique, ce dernier s'identifie naturellement avec l'espace de la géométrie Euclidienne. En effet, plus de deux millénaires après sa formalisation, l'espace Euclidien reste à l'époque le seul candidat susceptible de représenter l'espace physique, à tel point que le philosophe Emmanuel Kant, dans sa *Critique de la Raison pure* (1780) en fit l'exemple paradigmatique de connaissance synthétique *a priori*, qualifiant l'espace d'Euclide de "nécessité inévitable de la pensée".

Le changement de paradigme que constitua l'avènement de la géométrie Hyperbolique, suite aux travaux de C. Gauss, J. Bolyai et N. Lobachevsky, vint disputer ce monopole millénaire. Plusieurs possibilités s'ouvrirent alors quant à la description géométrique de

2. Notons que la forme moderne de ces caractères doit en réalité être attribuée à Leibniz qui a développé parallèlement à Newton le calcul infinitésimal et en a fixé les notations.

l'espace physique (*cf.* la célèbre exclamation de Boylai : "J'ai créé un nouveau monde à partir de rien. "), cette question ne pouvant dès lors plus être résolue par le simple raisonnement abstrait mais constituant désormais un problème d'ordre physique pouvant par là-même faire l'objet d'un test empirique [3].

Les quelques décennies qui suivirent la découverte de la géométrie Hyperbolique virent le développement d'autres géométries non-Euclidiennes, telles que les géométries Elliptique, Affine et Projective. En dehors des cercles mathématiques, l'émergence de ces nouveaux types de géométries provoqua un vif émoi dans la vie intellectuelle européenne, engendrant notamment dans l'Angleterre Victorienne de l'époque d'importants débats quant au remaniement de l'enseignement de la géométrie alors basée sur les *Éléments* [4].

Face à cette profusion de géométries nouvelles, Felix Klein formula en 1872 une synthése reposant sur une conception renouvelée de la géométrie envisagée comme l'étude des propriétés invariantes de l'espace sous un groupe de transformations, mettant ainsi l'accent sur les liens entre géométrie et théorie des groupes.

Il faudra attendre le début du vingtième siècle pour voir la révolution non-Euclidienne irriguer les théories physiques avec la formulation quadridimensionnelle de la relativité restreinte Einsteinienne par Hermann Minkowski (dans son essai de 1908 intitulé *Espace et Temps*). L'approche de Minkowski repose sur l'idée ancienne de combiner espace et temps dans une entité à quatre dimensions. Cependant, l'indépendance supposée des deux notions rendait auparavant la procédure artificielle. La géométrisation de l'espace-temps opérée par Minkowski permet ainsi de réinterpréter les principes de la relativité restreinte comme des théorèmes d'une nouvelle géométrie à quatre dimensions. Les trajectoires des particules libres sont alors décrites par des géodésiques de l'espace-temps plat quadridimensionnel.

Cette approche a connu le succès que l'on sait et de fait contient les germes de la théorie de la relativité générale. La théorie de la gravitation formulée par Einstein en 1915 constitue en effet la première géométrisation réussie d'une interaction fondamentale. Le contenu de la théorie de la relativité générale peut être schématiquement divisée en un versant cinématique (mouvement de la matière dans un espace-temps de géométrie fixée) et un autre dynamique (rétroaction de la matière sur la géométrie de l'espace-temps), les deux reposant sur le formalisme de la géométrie différentielle (pseudo)-Riemannienne. L'aspect cinématique de la relativité générale est similaire à celui de la formulation Minkowskienne de la relativité restreinte, à la différence que les particules libres y décrivent des géodésiques dans un espace-temps courbe. La courbure de l'espace-temps est elle-même engendrée par la

3. Lobachevsky suggéra ainsi de décider la géométrie de l'Univers à l'aide d'une expérience de mesure d'un "triangle stellaire" formé par l'étoile Sirius et deux positions de la Terre à six mois d'intervalle.

4. Parmi les protagonistes de cette controverse figure notamment Charles Lutwidge Dodgson, alias Lewis Carroll, qui publia en 1879 une pièce en défense des *Éléments*, intitulée *Euclid and his Modern Rivals*, mettant en scène le fantôme d'Euclide argumentant face à ses "rivaux modernes".

distribution de matière-énergie, ce qui constitue l'aspect dynamique de le théorie, encodée dans les équations de champ d'Einstein [5]. Einstein lui même, commentant ses équations de champ, exprima éloquemment son idée de la perfection géométrique en opposant l'expression "temple de marbre", pour désigner le géométrique tenseur de courbure d'Einstein, à celle d' "édifice en bois délabré", référant au tenseur énergie-impulsion [6].

À l'époque où Einstein compose sa théorie, la notion de connexion n'a pas encore acquis un statut propre et reste subordonnée à celle de structure métrique. Le transport parallèle des géodésiques décrivant les particules en chute libre s'identifie donc naturellement avec la notion de parallélisme étudiée par Christoffel, Ricci et Levi-Civita. Il faudra attendre les travaux d'Hermann Weyl en 1918 et ceux d'Elie Cartan sur les espaces fibrés, à partir de 1922, pour voir émerger une notion autonome de connexion.

Cette indépendance conquise par la notion de connexion va jouer un rôle fondamental dans l'entreprise de géométrisation de la théorie Newtonienne de la gravitation. Il est en effet significatif que la première description invariante de coordonnées de la gravitation de Newton ait été proposée par E.Cartan lui-même [11, 12] (1923).

En effet, contrairement à un préjugé tenace, la possibilité d'une géométrisation des phénomènes gravitationnels (*i.e.* de l'espace-temps) n'est pas l'apanage des théories dites relativistes mais existe tout aussi bien pour les théories dites nonrelativistes [7]. Indépendamment des travaux d'E.Cartan, Kurt Friedrichs introduit en 1927 (*cf.* [14]) l'utilisation d'une métrique dégénérée avec une connexion de Koszul compatible à torsion nulle. Du point de vue dynamique, Friedrichs montra que les équations de champ Newtoniennes dans le vide prennent la forme (tenseur de Ricci)= 0, exactement comme dans le cas relativiste.

Cette description géométrique de la gravitation Newtonienne (théorie dite de *Newton-Cartan* [8]) connut un renouveau d'intérêt à partir des années 1960, dont l'histoire déploie la liste (non-exhaustive) de noms suivante : Trautman [15], Havas [16], Dombrowski [17], Künzle [18], Ehlers [19], *etc.*

5. Pour résumer ces deux versants dans les mots du physicien américain John A. Wheeler : "Space tells matter how to move. Matter tells space how to curve. " (*cf.* [1] p.5)

6. Comme observé dans [2], cette distinction tranchée opérée par Einstein s'est vue quelque peu amendée suite aux travaux de divers auteurs [3, 4, 5, 6, 7, 8, 9] défendant le point de de vue selon lequel l'élégant et géométrique membre de gauche des équations d'Einstein peut être reconstruit en faisant le postulat physique que le (non géométrique) tenseur énergie-impulsion joue le rôle de source pour la gravitation (voir *e.g.* le Chapitre 3 du livre [10] pour une introduction pédagogique à ce point de vue.)

7. Plusieurs auteurs (dont Levy-Leblond, *cf.* [13]) ont commenté le caractère impropre d'une telle terminologie. En effet, les théories dites nonrelativistes encodent tout autant que celles dites relativistes le principe de relativité. Seule change l'expression des boosts (Lorentz *vs* Galilée).

8. Nous désignerons par Newton-Cartan un spectre de théories plus large que celui couvert par l'acception habituelle, référant notamment à des théories admettant des variétés dont les tranches d'espace sont courbes.

Gravitation relativiste et nonrelativiste

Le diagramme suivant synthétise les relations entre les différents acteurs du contenu cinématique de la relativité générale.

$$\begin{tikzcd} & \text{Métrique Lorentzienne} \arrow[dl, "1"'] \arrow[dr, "3"] & \\ \text{Connexion de Levi-Civita} \arrow[rr, "2"'] & & \text{Géodésiques} \end{tikzcd}$$

FIGURE 1 – Contenu cinématique de la relativité générale

Le rôle de structure métrique y est joué par une variété d'espace-temps munie d'une métrique (pseudo)-Riemannienne (ou Lorentzienne) *i.e.* un champ de formes bilinéaires non-dégénérées sur la variété. Cette structure métrique détermine (*cf.* Théorème 3.1.5) une unique connexion (au sens de Koszul) à torsion nulle qui lui est compatible, à savoir la connexion de Levi-Civita (flèche 1). Cette connexion munit donc la variété d'espace-temps d'une notion de parallélisme, permettant de définir une classe de courbes privilégiée sur la variété, les géodésiques (flèche 2). Une géodésique est ainsi définie comme une courbe dont le vecteur tangent reste en tout point parallèle à lui-même, eu égard au parallélisme de Levi-Civita. Les courbes géodésiques peuvent de manière équivalente être caractérisées comme les courbes qui extrémisent la distance entre deux points infiniment proches. L'équation caractéristique des géodésiques peut donc être obtenue comme équation du mouvement à partir d'une densité Lagrangienne construite uniquement en termes de la structure métrique (flèche 3). Les relations entre ces différentes structures sont exprimées de manière abstraite par le diagramme suivant :

$$\begin{tikzcd} & \text{Structure métrique} \arrow[dl, "1"'] \arrow[dr, "3"] & \\ \text{Connexion} \arrow[rr, "2"'] & & \text{Géodésiques} \end{tikzcd}$$

FIGURE 2 – Contenu cinématique des théories métriques de la gravitation

Il est intéressant de constater que le contenu cinématique de la gravitation Newtonienne peut également être décrit via un diagramme similaire, les différences vis-à-vis du cas relativiste étant dues au caractère dégénéré de la structure métrique Newtonienne (désignée ci-après par le terme de *structure Augustinienne*, *cf.* Définition 3.2.1). Une des conséquences les plus immédiates liée à la nécessité de faire usage d'une métrique dégénérée réside dans le fait qu'à une structure Augustinienne donnée peut être associée une *classe* de connexions de Koszul compatible à torsion nulle. Cette multiplicité de connexions invite à définir une structure plus riche (dénommée *structure Lagrangienne*, *cf.* Définition 3.2.40) permettant ainsi de rétablir l'unicité (flèche 1). La connexion Newtonienne (*cf.* Définition 3.2.33) ainsi définie pourvoit l'espace-temps Newtonien d'un parallélisme (différent de celui de Levi-Civita) permettant la définition de courbes auto-parallèles, similairement au cas relativiste. De telles courbes acquièrent ici l'interprétation de trajectoires dynamiques pour une classe très générale de systèmes holonomes[9] (*cf.* Section 1.3) et peuvent dès lors être dérivées via un principe variationnel construit à partir de la structure Lagrangienne (flèche 3).

$$\begin{tikzcd} & \text{Structure Lagrangienne} \arrow[dl, "1"'] \arrow[dr, "3"] & \\ \text{Connexion Newtonienne} \arrow[rr, "2"'] & & \text{Trajectoires dynamiques} \end{tikzcd}$$

FIGURE 3 – Contenu cinématique de la théorie de Newton-Cartan

Structures Newtoniennes plongées dans des structures Bargmanniennes

Jusqu'à présent, nous n'avons envisagé les structures Newtoniennes que dans leur caractérisation intrinsèque, ne faisant référence à des structures relativistes que pour les besoins de l'analogie. Ce type de comparaison peut cependant laisser au lecteur l'impression que les structures dites nonrelativistes sont en un sens moins naturelles que leurs avatars relativistes[10]. Un nouvel éclairage peut cependant être porté sur les structures Newtoniennes en envisageant ces dernières comme plongées dans des structures relativistes. Ce nouveau point de vue a le mérite d'apporter une justification de ces structures et de leurs propriétés en les important de structures relativistes usuelles. Ce point de vue prend sa source dans un article datant de 1928 dû à L.Eisenhart [20] dans lequel ce dernier établit que les trajectoires dynamiques d'un système mécanique nonrelativiste holonome à n degrés de liberté peuvent toujours être mises en correspondance bijective avec les géodésiques d'un espace-temps relativiste à $n+2$ dimensions doté d'une structure métrique particulière. Ainsi, les trajectoires dynamiques peuvent toujours êtres "remontées" vers des géodésiques (d'où la dénomination "Eisenhart lift") et inversement, toute géodésique peut se voir pro-

9. Un système dynamique à d degrés de liberté $x_1, \cdots, x_d$ soumis à n contraintes pouvant être exprimées sous la forme $f^i(x_1, \cdots, x_d, t) = 0$, avec $i \in \{1, \cdots, n\}$ est dit *holonome*. Les contraintes d'un système holonome dont l'énergie cinétique prend la forme standard $T = \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b$ peuvent toujours être résolues. Un tel système est donc équivalent à un système dépourvu de contraintes dont l'énergie cinétique prend la forme $T = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j + A_i q^i - U$.

10. Nous renvoyons au §12.5 du livre [1] pour une réflexion d'ordre épistémologique sur la naturalité des théories physiques dans leur formulation géométrique.

jetée vers une trajectoire nonrelativiste. La classe d'espace-temps permettant le Eisenhart lift est caractérisée par l'existence d'un champ vectoriel de genre lumière parallèle par rapport à la connexion de Levi-Civita. Cette classe de métriques s'était déjà vue décrite par M.Brinkmann [21] dans un contexte différent et reçut par la suite l'interprétation d'ondes gravitationnelles avec rayons parallèles, les rayons étant les courbes intégrales du champ vectoriel nul et parallèle. Cet important résultat de Eisenhart demeura en réalité largement confidentiel parmi les physiciens théoriciens pendant plusieurs décennies (à l'exception notable d'A.Lichnerowicz [22], sur laquelle nous reviendrons). Parmi les raisons possibles de ce désintérêt figure probablement le fait que la procédure consistant à associer à un système dynamique à n degrés de liberté un espace-temps de dimension $n+2$ peut sembler artificielle, en comparaison avec les espaces de dimensions respectivement n et $n+1$ que sont l'espace des configurations et l'espace-temps des configurations, plus naturellement associés à un tel système Lagrangien (*cf.* Section 1 de [23]).

Ce pont dressé entre physique nonrelativiste et espace-temps relativiste a été ensuite redécouvert de manière indépendante par C.Duval, G.Burdet, H.P.Künzle et M.Perrin dans un article de 1985 [24] (*cf.* également [25]) dans lequel les auteurs généralisent cette approche *à la* Eisenhart, dite "ambiante", afin de rendre compte des différents aspects cinématiques de la physique Newtonienne, tels que décrits dans le paragraphe précédent, comme plongés dans des variétés de Bargmann-Eisenhart [11].

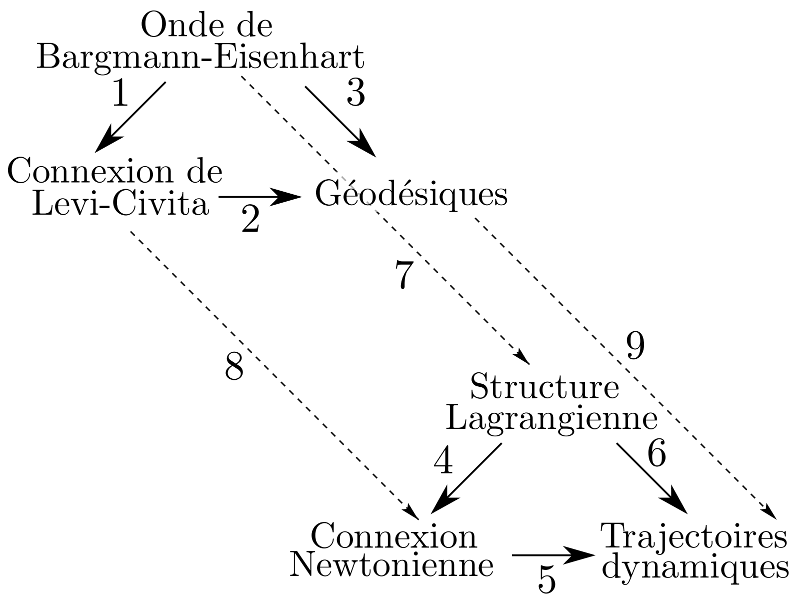


FIGURE 4 – Structures Newtoniennes plongées dans des structures Bargmanniennes

11. La référence à V.Bargmann [26] est justifiée par le rôle déterminant joué par l'extension centrale du groupe de Galilée (dit groupe de Bargmann), dans la construction de [24, 25].

Le diagramme précédent résume les différentes étapes relatives au plongement de structures Newtoniennes dans des structures Bargmanniennes. Comme indiqué précédémment, les variétés de Bargmann-Eisenhart (*cf.* Définition 2.1.1) sont des espace-temps relativistes possédant un champ de vecteur nul (dénommé *vecteur d'onde* par la suite), transporté parallèlement par la connexion de Levi-Civita associée à la métrique (*cf.* eq. (1.3.13) pour une expression explicite du tenseur métrique en coordonnées). L'idée principale de l'approche ambiante appliquée à la physique nonrelativiste est d'opérer une réduction dimensionnelle d'une variété de Bargmann-Eisenhart le long de cette direction de genre lumière. En cela, cette approche diffère du formalisme de Kaluza-Klein usuel dans lequel la réduction s'effectue le long d'une direction de genre spatial [12] ou encore de la réduction le long d'un vecteur de genre temps pour des espace-temps stationnaires. La variété obtenue comme quotient de la variété de Bargmann-Eisenhart par la direction nulle hérite ainsi une structure d'espace-temps Newtonien, ce qui peut s'apprécier aux différents niveaux de structure représentés sur le diagramme de la Figure 2.

Ainsi, la structure métrique de l'onde de Bargmann-Eisenhart projette sur la variété quotient en une structure Lagrangienne (flèche 7) tandis que la connexion de Levi-Civita associée à la structure métrique relativiste définit la connexion Newtonienne (flèche 8) correspondant à cette même structure Lagrangienne. Le Eisenhart lift compris comme une correspondance entre trajectoires dynamiques et géodésiques relativistes, est symbolisé par la flèche 9. De manière plus précise, on peut envisager le Eisenhart lift comme une composition des flèches 3 et 9, en ce sens que l'équation décrivant les trajectoires dynamiques nonrelativistes est obtenue par variation de la densité Lagrangienne construite à partir de la métrique relativiste. Le théorème d'Eisenhart assure que l'application $\xrightarrow{3}\xrightarrow{9}$ est bijective (à $p_u = m$ fixé, *cf.* Chapitre 1) pour la classe d'espace-temps considéré.

Généralisation

Revenons à présent sur l'apport d'A.Lichnerowicz au théorème d'Eisenhart. Ce dernier, dans son livre intitulé *Théories relativistes de la gravitation et de l'électromagnétisme* [22] et datant de 1955, généralisa la classe d'espace-temps relativistes permettant le Eisenhart lift. Un aspect important de cette généralisation consiste en cela que les trajectoires d'un système nonrelativiste donné peuvent désormais être "remontées" vers les géodésiques d'une classe infinie d'espace-temps. Pour poursuivre la formulation imagée du paragraphe précédent, on peut dire que l'application $\xrightarrow{3}\xrightarrow{9}$ n'est désormais plus injective pour cette classe élargie d'espace-temps.

Cette importante contribution de Lichnerowicz au formalisme ambiant soulève la ques-

12. De plus, le groupe de structure de la fibration $(\mathbb{R}, +)$ est non-compact, *cf.* Chapitre 4.

tion naturelle de déterminer dans quelle mesure il est possible de généraliser les différents niveaux d'intégration symbolisés sur le diagramme de la Figure 4 à cette classe élargie. Une telle étude, au-delà de son intérêt intrinsèque, peut être motivée en vue de ces potentielles applications, notamment dans le contexte de la dualité théorie de jauge/gravitation. En effet, depuis sa proposition en 1998 par J.Maldacena [27, 28], la correspondance [13] AdS/CFT a fait l'objet d'un intérêt sans cesse renouvelé et ce parmi un nombre croissant de communautés de physiciens. En particulier, il a été plus récemment suggéré que les techniques de la correspondance holographique pouvaient se voir appliquées à des systèmes nonrelativistes, avec l'espoir de permettre ainsi de décrire des systèmes de physique de la matière condensée. Parmi les premiers candidats susceptibles d'une description holographique figurent notamment les supraconducteurs à $2 + 1$ dimensions [29] ainsi que les gaz unitaires d'atomes froids [30, 31]. Ce dernier système se distingue par le fait que l'espace-temps de fond de la théorie vivant dans le "bulk" diffère de l'espace-temps d'AdS usuel mais en est une déformation choisie pour posséder un groupe d'isométries nonrelativiste (*i.e.* le groupe de Schrödinger), d'où la dénomination d'espace-temps de Schrödinger [32, 33]. Cette nouvelle approche à l'holographie a suscité un renouveau d'intérêt quant aux symétries nonrelativistes et à leur description géométrique [34]. Dans ce contexte, la généralisation proposée par Lichnerowicz acquiert une nouvelle actualité. Il est en effet possible de montrer que cette classe élargie de variétés permettant le plongement de structures nonrelativistes contient notamment les espaces-temps d'Anti-de Sitter et de Schrödinger (*cf.* Section 2.3), espaces-temps qui n'étaient pas contenus dans la classe étudiée par Eisenhart. Ce résultat suggère d'appliquer l'approche ambiante à la dualité holographique esquissée dans les travaux [30, 31], de manière à obtenir une dualité holographique purement nonrelativiste (*i.e.* reliant une théories nonrelativiste de la gravitation dans le bulk à une théorie nonrelativiste de la matière condensée sur le bord, voir *e.g.* [35, 36, 37]). Dans cet ordre d'idées, une correspondance entre le gaz unitaire de Fermi et une théorie de spin-élevés a récemment été proposée [38, 39].

Depuis son introduction dans [24, 25], l'approche par plongement s'est révélée pertinente pour traiter un large éventail de problèmes nonrelativistes, tels que l'électrodynamique de Chern-Simons [40, 41, 42], la dynamique des fluides [43, 44, 45], la cosmologie de Newton-Hooke [46], les symétries de Schrödinger [32, 34], le théorème de Kohn [47, 48, 49], les symétries cachées [50, 51, 52], *etc.* Au-delà de sa pertinence dans le contexte de la correspondance AdS/CFT, un élargissement du formalisme ambiant de [24, 25] pourrait ainsi permettre de généraliser certains de ces résultats en s'appuyant sur la liberté supplémentaire présente dans la classe d'espace-temps étudiée par Lichnerowicz.

Sur le plan conceptuel, l'approche suivie dans [24, 25] repose de manière cruciale sur

13. On utilisera les acronymes anglo-saxons AdS pour l'espace-temps Anti de-Sitter et CFT pour théorie conforme des champs.

l'utilisation du groupe de Bargmann, *i.e.* l'extension centrale du groupe de Galilée. Les variétés de Bargmann-Eisenhart sont ainsi décrites comme des G-structures pour le sous-groupe de Bargmann homogène (*cf.* également [53]). L'approche de Duval *et al.* tire en effet parti de ce qui peut être désigné comme l'équivalent au niveau théorie des groupes du formalisme ambiant, à savoir le fait que certains groupes de symétrie nonrelativistes (*e.g.* les groupes de Bargmann [26] et Schrödinger [54]) peuvent être plongés dans leurs homologues relativistes (groupes de Poincaré [55] et conforme [56], respectivement). Les groupes nonrelativistes peuvent être ainsi obtenus par ce qui est l'analogue en théorie des groupes d'une réduction dimensionnelle de genre lumière, à savoir comme sous-groupes préservant une certaine direction de type nul. Cette relation peut également être appréciée au niveau des équations d'onde en espace-temps plat (*cf.* [57]).

Le leitmotif consistant à importer des résultats et structures provenant de la théorie des groupes vers des structures géométriques est réminiscent du point de vue que porte Cartan sur la géométrie différentielle, approche qui a connu depuis la fin des annés 90 un regain d'intérêt (à la fois dans la littérature mathématique et de physique théorique) suite notamment à la parution de la première introduction moderne au sujet (*cf.* [58]). En particulier, il a été suggéré [59, 60, 61] que la géométrie de Cartan constitue la formulation appropriée à la description de théories de la gravitation comme théories de jauge. En effet, il est bien connu dans la littérature de physique théorique que la formulation rigoureuse des théories de jauge dont le groupe de transformation est semi-simple (théories de type Yang-Mills) requiert l'introduction d'un fibré principal ayant pour base la variété d'espace-temps et muni d'une connexion d'Ehresmann [14] prenant valeur dans le groupe considéré. En revanche, cette formulation est impropre à décrire la théorie de la relativité générale, cette dernière ne pouvant être vue comme une théorie de type Yang-Mills (*cf.* *e.g.* [63] ou l'appendice A de [64]). Cependant, la théorie d'Einstein, dans la formulation qu'en donnent S.W. MacDowell et F. Mansouri [65] (*cf.* également [66]) peut être interprétée comme une théorie dynamique d'une connexion de Cartan pour une géométrie modelée sur la paire de Klein $(\mathrm{SO}\,(d,2), \mathrm{SO}\,(d,1))$, *cf.* Section 5.1, donnant ainsi une interprétation géométrique à la brisure de l'invariance de jauge de la théorie, du groupe d'isométrie d'AdS vers le groupe de Lorentz (*cf.* [60]). Cette approche s'est également révélée féconde pour clarifier le statut géométrique de la formulation donnée par Witten [67] de la gravité à $2+1$ dimensions comme théorie de Chern-Simons [59].

La seconde partie de ce manuscrit est ainsi construite autour de l'idée de "rendre Newton-Cartan à Cartan" en embrassant le postulat selon lequel les géométries de Cartan projettent un nouvel éclairage sur les structures nonrelativistes, à la fois dans leur définition

14. Charles Ehresmann fut l'étudiant de Cartan et a poursuivi la quête de ce dernier de conférer un statut réellement géométrique à la notion de connexion. Dans [62], Ehresmann donna à la fois une définition rigoureuse des connexions *à la* Cartan et des connexions qui portent désormais son nom.

intrinsèque et dans leur relation avec des structures relativistes au travers du formalisme ambiant.

Plan

Le présent manuscrit se divise en deux parties. La première propose une généralisation du formalisme ambiant de Duval *et al.* [24, 25] à la classe d'espace-temps étudiés par Lichnerowicz [22] (dites *ondes Platoniciennes*) permettant ainsi une formulation géométrique générale du théorème d'Eisenhart-Lichnerowicz.

Dans le Chapitre 1, on rappellera le théorème d'Eisenhart-Lichnerowicz pour la classe des ondes Platoniciennes, d'abord dans son cadre Lagrangien usuel, puis on proposera une formulation Hamiltonienne de ce théorème pour cette classe élargie (*cf.* également [50, 51, 52] pour une formulation Hamiltonienne du Eisenhart lift pour la classe des ondes de Bargmann-Eisenhart). Nous défendrons alors l'idée selon laquelle cette perspective Hamiltonienne est la mieux à même de capturer la simplicité du lift d'Eisenhart-Lichnerowicz, dont l'élégance est habituellement dissimulée derrière la complexité des équations du mouvement Lagrangiennes. Une présentation heuristique de l'approche ambiante sera ensuite proposée en filant l'analogie suggérée dans [68] entre le formalisme ambiant et la célèbre allégorie de la Caverne de Platon [69]. L'approche ambiante se verra ensuite étendue des particules classiques aux particules quantiques ; l'équation de Schrödinger sur un espace courbe sera ainsi obtenue par réduction dimensionnelle à partir de l'équation de Klein-Gordon pour un champ scalaire libre sur une onde Platonicienne.

Le Chapitre 2 sera consacré à une analyse des propriétés géométriques des ondes gravitationnelles et des structures nonrelativistes vivant sur l'espace de leurs rayons (*i.e.* l'espace quotient par la direction de genre lumière, dénommé *écran Platonicien* par la suite). Un intérêt particulier sera porté aux classes dites de Kundt, Platon et Bargmann-Eisenhart et aux structures métriques nonrelativistes qu'elles induisent. Une nouvelle définition géométrique des ondes Platoniciennes sera proposée et comparée avec d'autres caractérisations existantes. Il sera par la suite fait usage de cette nouvelle définition afin de généraliser à cette classe certains résultats connus pour les ondes de Bargmann-Eisenhart concernant d'une part leurs propriétés globales et causales, et d'autres part leurs invariants de courbure scalaires.

Le Chapitre 3 délaissera provisoirement l'approche ambiante pour se consacrer à l'étude des structures nonrelativistes, envisagées d'un point de vue purement intrinsèque. Une première partie comprendra un état de l'art dont l'ambition sera d'être pédagogique sans toutefois concéder à une certaine exigence de rigueur dans l'exposé. L'accent sera mis notamment sur le contraste entre de telles structures et leur homologues relativistes. Reflétant la littérature, on détaillera dans un premier temps les variétés nonrelativistes dont la structure métrique comprend une 1-forme d'horloge fermée (structure dite *Augustini-*

enne) et les notions de parallélisme associées [15] (connexions *Galiléennes* et *Newtoniennes*). Ces dernières seront également abordées sous une forme légèrement moins standard dans une formulation reposant sur l'existence de champs d'observateurs irrotationnels (structure dite *Lagrangienne*, *cf.* [23]). Dans une seconde partie, on s'intéressera à la généralisation de certaines de ces structures pour des variétés nonrelativistes dont la 1-forme d'horloge satisfait le critère de Frobenius (structure dite *Aristotelicienne*). Deux notions concurrentes de parallélisme, reposant sur la généralisation au cas Aristotelicien des connexions Newtoniennes dans leur formulation respectivement usuelle et Lagrangienne, se verront proposées et discutées à la lumière du théorème d'Eisenhart-Lichnerowicz.

Outre une reformulation dans le langage des fibrés principaux des résultats du Chapitre 2 liés aux plongement de structures métriques nonrelativistes dans une onde gravitationnelle, le lecteur trouvera dans le Chapitre 4 un examen approfondi de la notion de parallélisme abordée d'un point de vue ambiant. Après avoir passé en revue les résultats de [24, 25] relatifs au plongement d'une variété Newtonienne dans une onde de Bargmann-Eisenhart, on étendra cette construction à la classe des ondes Platoniciennes. Un aspect important de cette généralisation réside dans la non-unicité de la connexion nonrelativiste induite par projection du parallélisme de Levi-Civita ambiant sur l'écran Platonicien. On isolera parmi les choix possibles deux prescriptions différentes, permettant d'apporter un éclairage ambiant sur les connexions introduites en fin de Chapitre 3 et discutera de la pertinence de cette construction quant à la géométrisation du théorème d'Eisenhart-Lichnerowicz.

La seconde partie de ce manuscrit est consacrée à l'étude des géométries de Cartan et à leur intérêt quant à la formulation des théories de gravitation nonrelativistes. Le Chapitre 5 se présente comme une introduction aux géométries de Klein et à leur généralisation par Cartan. Cette présentation générale sera émaillée d'exemples issus de la géométrie (pseudo)-Riemanienne, envisagée comme géométrie de Cartan structurée par le groupe de Poincaré. Une attention particulière sera portée à la façon dont des structures algébriques naturelles d'un point de vue de la théorie des groupes induisent des structures géométriques tout aussi naturelles sur la variété d'espace-temps.

Le Chapitre 6 se proposera d'appliquer ce leit-motiv dans le but de réinterpréter *à la* Cartan certains pans bien connus de la physique nonrelativiste. Ainsi, les variétés Galiléennes seront envisagées comme espaces de base d'une géométrie de Cartan admettant comme groupe de structure le groupe de Galilée. Cette approche permettra notamment d'examiner l'origine algébrique de la notion de *champ d'observateurs* et de *Milne boost* (*cf.* Définitions 3.2.5 et 3.2.10). Les variétés Newtoniennes seront ensuite caractérisées comme espace de base d'une géométrie de Cartan-Galilée plongée dans une géométrie (non réduc-

15. On restreindra l'analyse aux connexions à torsion nulle.

tive) de Cartan-Bargmann, construction qui nous permettra ainsi de dériver naturellement la condition classique de contrainte sur le tenseur de courbure (dite de *Duval-Künzle*) comme condition d'involutivité de la distribution définissant ledit plongement. Outre son rôle dans la définition de structures intrinsèquement nonrelativistes, le groupe de Bargmann se révélera également pertinent en regard de la formulation *à la* Cartan du formalisme ambiant. Ainsi, les ondes de Bargmann-Eisenhart se verront caractérisées comme espace de base d'une géométrie (réductive) de Cartan-Bargmann, permettant notamment de réinterpréter la contrainte de *Duval-Künzle* comme condition de torsion nulle.

Par souci de lisibilité, le plan de ce manuscrit a été agencé de manière à ménager une progression en pente douce en termes d'abstraction et de géométrisation [16]. Ainsi, les deux premiers Chapitres ont très largement recours à la notation indicielle ainsi qu'à l'utilisation d'un système de coordonnées, dit de Brinkmann. De plus, seules des notions élémentaires de géométrie Riemannienne y sont mobilisées afin d'établir les résultats présentés. Du point de vue nonrelativiste, seules les notions de structures métriques y sont examinées, réservant l'étude de la notion de parallélisme, d'un point de vue intrinsèque et ambiant, respectivement aux Chapitres 3 et 4. Le Chapitre 4 fait quant à lui usage de la géométrie des fibrés principaux en se restreignant cependant à l'utilisation de connexions de Ehresmann. Peut-être moins courante est l'utilisation qui est faite au Chapitre 6 des connexions de Cartan, mais nous espérons que le lecteur non déjà familier avec ces notions trouvera dans le Chapitre 5 matière à susciter son intérêt pour ces géométries.

En termes d'originalité du contenu, les Chapitres 1 et 2 reproduisent de manière *ad verbatim* le texte de l'article [70]. Le reste de ce manuscrit est quant à lui original et constituera la matière de l'article [71], à paraître.

16. Le lecteur restant seul juge de la mesure dans laquelle cet objectif a été atteint.

Perspectives

L'introduction qui précède a mis l'accent sur la restriction du cadre de la présente étude au contenu cinématique des théories de gravitation. Parmi les perspectives naturelles s'inscrivant dans la continuité de ladite étude figure donc une analyse des équations de champs gravitationnels, autrement dit du contenu dynamique de ces théories. En ce qui concerne l'approche ambiante, Duval *et al.* [24, 25] ont montré comment les équations de champ d'Einstein pour la classe d'espace-temps de Bargmann-Eisenhart se ramenaient, après réduction dimensionnelle, aux équations de champ de Newton-Cartan. Il serait dès lors intéressant de reproduire ce résultat pour la classe des ondes Platoniciennes, vues comme conformément Bargmann-Eisenhart et de comparer les équations de Newton-Cartan généralisées obtenues avec l'analyse de Julia *et al.* [72].

Une seconde limitation imposée au cadre de notre analyse consiste à ne prendre en considération que des géométries à torsion nulle. Une telle restriction reste assez naturelle quand l'on s'intéresse à des structures métriques nonrelativistes dont l'horloge est fermée (structure Augustinienne) puisque dans ce cas, il existe des connexions à torsion nulle compatible avec la structure métrique, de manière semblable au cas relativiste. Cependant, l'introduction de connexions à torsion non-nulle gagne en pertinence lorsque l'on considère des structures Aristoteliciennes, pour lesquelles les conditions de torsion nulle et de compatibilité à la structure métrique sont mutuellement exclusives. De telles considérations suggèrent naturellement une extension du formalisme ambiant étudié ici à des ondes Platoniciennes avec torsion. Une telle généralisation pourrait permettre d'étendre le champ des applications possibles du formalisme ambiant à l'holographie et à la matière condensée, telles qu'esquissées dans l'introduction, dans la mesure où de telles théories de Newton-Cartan avec torsion (au niveau nonrelativiste intrinsèque) ont récemment fait leur apparition pour décrire les symétries de systèmes de matière condensée exhibant l'effet Hall quantique fractionnaire (*cf.* [73, 74]).

L'articulation du présent manuscrit en deux parties suggère de généraliser les résultats préliminaires obtenus au Chapitre 6 afin de proposer une description dans le formalisme de Cartan des structures "Platoniciennes" nonrelativistes et relativistes discutées aux Chapitres 3 et 4, respectivement. Outre la gravitation de Newton-Cartan, l'approche à la Cartan pourrait projeter un éclairage nouveau sur d'autres théories nonrelativistes de la gravitation. Dans un célèbre article datant de 1968 [75], Bacry et Levy-Leblond entreprirent la classification des algèbres de Lie dites cinématiques, à savoir les algèbres encodant les symétries infinitésimales d'une particule libre. Leur classification distingue les algèbres dites relativistes (Poincaré, (A)dS) des algèbres nonrelativistes. Chacune de ces algèbres est associée à un groupe de Lie et définit un espace homogène. Outre l'algèbre de Galilée et son extension centrale (le groupe de Bargmann) figurent parmi les algèbres nonrelativistes celles de Newton-Hooke et de Carroll. Les algèbres de Newton-Hooke peuvent être vues comme un équivalent nonrelativiste des algèbres d'(A)dS. D'un point de vue ambiant, l'espace homogène de Newton-Hooke peut être obtenu par réduction dimensionnelle de genre lumière d'une Hpp-wave dont le groupe d'isométries est précisément l'extension centrale du groupe de Newton-Hooke, comme montré dans [46]. Récemment, le groupe de Carroll [76] a connu un renouveau d'intérêt suite aux travaux [77, 78, 79] (*cf.* aussi [80, 81]) montrant d'une part la relation de dualité que ce groupe entretient avec le groupe de Bargmann [77], et plus important encore, le lien avec le groupe de Bondi-Metzner-Sachs (BMS), ce dernier pouvant être caractérisé comme extension conforme du groupe de Carroll [78, 79]. Il serait dès lors intéressant d'appliquer la procédure décrite au Chapitre 6 aux algèbres de Carroll et Newton-Hooke et de comparer les structures obtenues avec celles dérivant des algèbres de Galilée/Bargmann. Enfin, une application du formalisme de Cartan aux structures conformes nonrelativistes (*cf.* [82, 34]) pourrait apporter un éclairage nouveau sur lesdites structures.

La gravitation de Newton-Cartan a récemment connu un regain d'attention dans le contexte des théories de supergravité (*cf.* [83, 84, 85]). L'approche adoptée dans ces différents travaux repose sur une procédure consistant à jauger une super-algèbre nonrelativiste (algèbre de super-Bargmann) puis d'imposer des conditions de contrainte, générant ainsi une théorie de supergravité de Newton-Cartan. Cette procédure est identique à celle utilisée dans [86] permettant de retrouver la théorie de Newton-Cartan en jaugeant l'algèbre de Bargmann. Au Chapitre 6 du présent travail se trouve défendue l'idée selon laquelle l'approche *à la* Cartan de la géométrie différentielle constitue le formalisme géométrique sous-tendant la procédure de [86]. Une telle géométrisation de la procédure de jauge en termes de géométries de Cartan, outre son attrait esthétique en tant que structure globale et indépendante des coordonnées, fournit notamment une interprétation naturelle des conditions de contrainte (comme conditions d'involutivité ou de torsion nulle, *cf.* Chapitre 6). À la lumière des travaux [83, 84, 85] émerge alors l'hypothèse naturelle d'une généralisation possible du formalisme décrit aux Chapitres 5 et 6 pour une connexion prenant valeur, non plus dans une algèbre de Lie, mais dans une super-algèbre nonrelativiste. De plus, la pertinence de la description *à la* Cartan des structures nonrelativistes se révèle en outre dans le traitement qu'elle permet de l'approche ambiante à ces théories. Un tel éclairage porté sur les théories de supergravité nonrelativistes, envisagées comme plongées dans des théories de supergravité usuelles permettrait ainsi d'en clarifier certains aspects à la lumière de leurs avatars relativistes.

Part I

Embedding nonrelativistic structures inside a gravitational wave

Chapter 1

Nonrelativistic dynamical trajectories as geodesic motions

In this Section, we start by introducing our notations and conventions. Then, we present the old results of Eisenhart [20] and Lichnerowicz [22], firstly, by reviewing the suggestive analogy proposed by Minguzzi between the null dimensional reduction and the allegory of the cave, secondly, by motivating the form of the ambient metrics as an extension of some class of nonrelativistic Lagrangians and, thirdly, by checking explicitly that the null dimensional reduction of the geodesic equations for a specific class of spacetimes in D dimensions boils down to the Euler-Lagrange equations of some holonomic dynamical systems of $d = D - 2$ degrees of freedom. However, this direct check in the Lagrangian framework (similar to the original proofs [20, 22]) is slightly cumbersome and partially obscures the simple mechanism behind the Eisenhart-Lichnerowicz Theorem. On the contrary, in the Hamiltonian formulation this mechanism becomes more transparent. Since the Hamiltonian version seems not to have been discussed in detail yet in the literature for the most general class (*cf.* however [50, 51, 52] for a Hamiltonian formulation of the Eisenhart lift for Bargmann-Eisenhart waves), it is presented in the last subsection.

1.1 Notations and conventions

We will use the "mostly plus" convention for the signature of Lorentzian spacetimes. The nonrelativistic spacetime will be a manifold of dimension n foliated by spatial hypersurfaces which are Riemannian manifolds of dimension $d = n - 1$. This manifold will be embedded inside an ambient relativistic spacetime of dimension $D = n + 1$. Minuscule greek indices $\mu, \nu, ...$ will denote "world" (holonomic) ambient indices while minuscule latin indices $a, b, ...$ will denote "tangent" (anholonomic) ambient indices, both taking $D = n+1$

values $(0, 1, 2, \ldots, D-1)$. Minuscule latin indices as $i, j, \ldots$ will denote (world or tangent) spatial indices taking $d = n-1$ values $(1, 2, \ldots, d)$. When it will be pertinent, one introduces the Cartesian coordinates $\vec{x} = (z, \vec{y})$ on Euclidean space $\mathbb{R}^d$.

1.2 Basic heuristics of the ambient approach

> *οι τοιουτοι ουκ αν αλλο τι νομιζοιεν το αλητες η τας των σκεναστων σκιας.*
>
> *To them, I said, the truth would be literally nothing but the shadows of the images.*
>
> – Plato, *The Republic*, Book VII (360 BCE)

Before introducing the technical details of the null dimensional reduction, the key ideas will be presented pictorially by pursuing the entertaining analogy proposed by Minguzzi between the ambient approach and the allegory of the cave [68].

The allegory of the cave was presented by Plato in his celebrated work "The Republic" as an illustration of his theory of Forms [69]. Prisoners are chained in the middle of a cave. They face a blank wall; behind them is a fire. They watch shadows projected on the wall in front of them by objects which move behind them and which they cannot see. In the allegory, the two-dimensional shadows represent material phenomena that can be perceived while the three-dimensional objects correspond to Plato's ideal Forms. According to Platonism, the ultimate reality is the world of Forms (3D objects), while Phenomena (2D images) are mere illusions because of the incomplete knowledge of mankind (prisoners). Leaving aside these philosophical views and focusing on our topic, the allegory of the cave provides an ancient example of "lightlike" dimensional reduction where objects are projected on a codimension-one manifold along light rays. [1] The analogy between the allegory of the cave and the ambient approach is even closer (Table 1.1): consider an ambient spacetime (playing the role of the cave in the allegory) on which a gravitational wave propagates and to which corresponds a congruence of graviton worldlines (replacing the light rays emitted by the fire). Physicists detect the corresponding gravitons on a screen (the wall where photons are projected in the allegory). [2] This projection of ambient events on the

1. In a sense, linear perspective in graphical arts is an even simpler instance of "lightlike" dimensional reduction, where three-dimensional objects are represented on a two-dimensional surface via projection along visual light rays. However, this example is not as useful for illustrating our purpose because linear perspective is static while time plays a crucial role in the ambient construction.

2. The switch from the gravitational wave to the graviton description is simply understood by applying the standard rules of translation (between wave and particle language) from geometric optics where the

Allegory of the cave	Ambient approach
Cave	Ambient spacetime
Wall	Screen
Light rays	Graviton worldlines
Shadows	Nonrelativistic physics

Table 1.1: Analogy: Allegory of the cave / Ambient approach

screen along gravitational rays is the most concrete way of formulating the null dimensional reduction considered in this work. The main lesson from the ambient approach is that the relativistic spacetime and the particle trajectories appear nonrelativistic when read on the screen. In this sense, nonrelativistic structures are mere shadows of relativistic ones.

In order to present the heuristics behind this mathematical fact, notice that the screen registers the following events: absorption or emission of a graviton by the screen. These events are encoded via the position on the screen and the instant of the intersection. The description of the screen worldvolume (*i.e.* the time evolution of the screen) via these coordinates already suggests that the former might be endowed with a natural structure of (codimension-one) spacetime. What is more remarkable is that this structure is nonrelativistic and that the shadows on the screen from ambient geodesics have a natural interpretation as dynamical trajectories of nonrelativistic particles.

1.3 Nonrelativistic Lagrangian

Consider a smooth manifold with coordinates (t, x^i) and the most general Lagrangian that is a polynomial of degree two in the velocities $\dot{x}^i = dx^i/dt$:

$$L(t,x,\dot{x}) \quad = \quad \frac{1}{2}\,\bar{g}_{ij}\,(t,x)\,\dot{x}^i\dot{x}^j + \bar{A}_i\,(t,x)\,\dot{x}^i - \bar{V}\,(t,x) \tag{1.3.1}$$

where $\bar{g}_{ij}$ is sometimes called the mass matrix. In order to avoid ghosts and constraints, we require the kinetic term $\frac{1}{2}\,\bar{g}_{ij}\left(t,x^i\right)\dot{x}^i\dot{x}^j$ to be a positive-definite quadratic form in the velocities. A dynamical system described by (1.3.1) can always be interpreted as describing the motion of a charged particle minimally coupled to an electromagnetic field through the vector potential $\bar{A}_i$ and the scalar potential $\bar{V}$, called "effective" potential in the following, and moving on a Riemannian manifold with metric $\bar{g}_{ij}$.

propagation of wavefronts is equivalently described by its orthogonal rays, which can be interpreted as worldlines.

Leaving aside this interpretation, this class of Lagrangians corresponds to the most general holonomic dynamical system obeying d'Alembert's principle with external forces $F_i = \bar{F}_{ij}\dot{x}^j + \bar{F}_i$ at most linear in the velocity satisfying the two further requirements: the linear part $\bar{F}_{ij}\dot{x}^j$ in the velocity of the external force does not develop any power ($\bar{F}_{ij}\dot{x}^i\dot{x}^j = 0 \Leftrightarrow \bar{F}_{(ij)} = 0$ [3]) and derives from a vector potential ($\bar{F}_{ij} = 2\partial_{[i}\bar{A}_{j]}$) while the part independent of the velocity derives from a scalar potential ($\bar{F}_i = -\partial_t\bar{A}_i - \partial_i\bar{V}$). The vector and effective potentials may depend on time. The Lorentz force is indeed the perfect example of such an external force. For later purpose, let us emphasise that the holonomic coordinates x^i of a given holonomic system are only defined up to a reparameterisation

$$\begin{aligned} x^i &\rightarrow x'^i = x'^i(t,x) \\ t &\rightarrow t' = t \end{aligned} \tag{1.3.2}$$

which preserves the general form of (1.3.1), but redefines the various coefficients $\bar{g}_{ij}$, $\bar{A}_i$ and $\bar{V}$.

Let us emphasise that the gift of the Lagrangian (1.3.1) defines a nonrelativistic spatial metric on the manifold labeled by the coordinates (t, x^i). In other words, the mass matrix $\bar{g}_{ij}$, being positive definite, provides a collection of rulers at any event. As the notion of a nonrelativistic spacetime necessitates absolute rulers *and* clocks, this motivates the introduction of a collection of clocks, equivalent to the gift of a function $\Omega(t,x) > 0$ specifying the unit of time at each point of spacetime. The lapse $d\tau' = m\,d\tau$ of local time τ' measured by the local clock (along a trajectory) corresponding to the lapse dt of absolute time t is:

$$d\tau' = \Omega(t,x)\,dt = m\,d\tau, \tag{1.3.3}$$

where the constant m is introduced by analogy with affine parameters (which are also defined up to a multiplicative constant $\tau' = m\,\tau$) and will acquire soon the interpretation of a nonrelativistic mass.

Since our goal is to relate the Lagrangian (1.3.1) to the geodesic equation for some spacetime, let us stress the similarities and differences of such an action principle with the quadratic action principle for a geodesic. Suggestively, one can rewrite the action

$$S[\,x^i\,] = m\int L(t,x,\dot{x})\,dt \tag{1.3.4}$$

corresponding to the nonrelativistic Lagrangian (1.3.1) in terms of the local time along the

3. Curved (respectively, square) brackets over a set of indices denote complete (anti)symmetrisation over all these indices, with weight one, *i.e.* $S^{(\mu_1\cdots\mu_r)} = S^{\mu_1\cdots\mu_r}$ and $A^{[\mu_1\cdots\mu_r]} = A^{\mu_1\cdots\mu_r}$ respectively for S and A totally (anti)symmetric tensors.

trajectory as

$$S[\,x^i\,] = \int \Omega \left(\frac{1}{2}\,\bar{g}_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + \bar{A}_i\frac{dx^i}{d\tau}\frac{dt}{d\tau} - \bar{V}\frac{dt}{d\tau}\frac{dt}{d\tau}\right) d\tau\,. \tag{1.3.5}$$

where eq.(1.3.3) has been used. With the classical action (1.3.4) being defined up to a multiplicative constant, the factor m has been introduced for later purposes. Notice that the case $m = 0$ is special and corresponds to nondynamical trajectories in the sense that eq.(1.3.3) implies $dt = 0$ and so the curve $\big(t, x^i\,(\tau)\big)$ is at fixed t. Moreover, the action (1.3.5) becomes $S[\,x^i\,] = \frac{1}{2}\int \Omega\,\bar{g}_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}d\tau$ which has the form of a quadratic geodesic action for the metric $g_{ij} = \Omega\,\bar{g}_{ij}$.

The action (1.3.5) looks like the quadratic action for a geodesic in the spacetime described by the line element:

$$\begin{aligned} ds^2_{(n)} &= \Omega\left(\bar{g}_{ij}dx^i dx^j + 2\,\bar{A}_i\,dx^i dt - 2\,\bar{V}dt^2\right) \\ &= g_{ij}dx^i dx^j + 2\,A_i\,dx^i dt - 2\,V dt^2\,. \end{aligned} \tag{1.3.6}$$

However, an important discrepancy between (1.3.5) and the action principle for a geodesic corresponding to the line element (1.3.6) is that the parameter τ is not an affine parameter since its normalisation is not defined in terms of the metric defined by (1.3.6) but simply as

$$\Omega\frac{dt}{d\tau} = m\,. \tag{1.3.7}$$

Although the right-hand side (1.3.6) can naïvely be interpreted as a line element on the nonrelativistic n-dimensional spacetime, this metric has actually no definite signature since there is no *a priori* sign constraint on the potential V (which might even be vanishing). Nevertheless, the gift of a Lagrangian of degree two in the velocities and of a time unit is equivalent to the gift of an indefinite line element of spacetime. However, a nonrelativistic spacetime has a somewhat weaker structure: it is rather defined only by the clocks $\Omega(t,x)\,dt$ and by the rulers encoded in the spatial metric $d\ell^2 = g_{ij}(t,x)dx^i dx^j$ on the spatial leaves t=const.

In order to lift the dynamical trajectories ($m \neq 0$) to geodesics of an ambient spacetime, the crucial ingredient is to add the value of the action as an extra coordinate. More precisely, we introduce a coordinate u proportional to the action and to the local time τ such that the infinitesimal variation of the action (1.3.4) along a trajectory is equal to

$$du = -L\,dt - \frac{M^2}{m}\,d\tau. \tag{1.3.8}$$

The minus sign and normalisation have been chosen for later convenience. By making use of the relations (1.3.1) and (1.3.8), the line element (1.3.6) is equal to:

$$\Omega \left(\bar{g}_{ij} dx^i dx^j + 2\,\bar{A}_i\,dx^i dt - 2\,\bar{V} dt^2\right) = -2\Omega\,dtdu - 2M^2 d\tau^2. \tag{1.3.9}$$

The main idea behind the Eisenhart lift (in Lagrangian terms) is to make use of (1.3.7) in order to reinterpret this relation as expressing the fact that τ is an affine parameter along a geodesic in an ambient spacetime of coordinates $x^\mu \equiv (u, t, x^i)$ and suitable metric $g_{\mu\nu}$. More precisely, we want to rewrite (1.3.9) as the relation $g_{\mu\nu} dx^\mu dx^\nu = -M^2 d\tau^2$ where the constant $\left|M^2\right|$ stands for the ambient velocity norm squared. We will check that eq. (1.3.7) simply arises as an equation of motion. We should stress that there is a large ambiguity in reading off the ambient metric from (1.3.9) when the geodesics are not lightlike $\left(M^2 \neq 0\right)$. More precisely, the relation (1.3.9) can be rewritten as a normalisation condition for the affine parameter τ:

$$\Omega\,(t,x) \left[2\,dt \left(du + \bar{A}_i\,(t,x)\,dx^i - \bar{U}\,(t,x)\,dt\right) + \bar{g}_{ij}\,(t,x)\,dx^i dx^j\right] = -M^2\,d\tau^2 \tag{1.3.10}$$

if we define

$$\bar{U} = \bar{V} - \frac{1}{2}\frac{M^2}{m^2}\,\Omega\,. \tag{1.3.11}$$

In order to distinguish them, the potential $\bar{V}$ will be referred to as *effective potential* while the term *scalar potential* will be reserved to designate $\bar{U}$. If the geodesic is lightlike, then $M^2 = 0$ and thus $\bar{U} = \bar{V}$. The left-hand side of (1.3.10) can be interpreted as the ambient line element

$$\begin{aligned} ds^2 &= \Omega\,(t,x) \left[2\,dt \left(du + \bar{A}_i\,(t,x)\,dx^i - \bar{U}\,(t,x)\,dt\right) \,+\, \bar{g}_{ij}\,(t,x)\,dx^i dx^j\right] \\ &= 2\,\Omega\,(t,x)\,dtdu + 2\,A_i\,(t,x)\,dtdx^i - 2\,U\,(t,x)\,dt^2 \,+\, g_{ij}\,(t,x)\,dx^i dx^j \end{aligned} \tag{1.3.12}$$

The ambient metric g is conformally equivalent to the metric $\bar{g}$ with line element

$$d\bar{s}^2 = 2\,dt \left(du + \bar{A}_i\,(t,x)\,dx^i - \bar{U}\,(t,x)\,dt\right) \,+\, \bar{g}_{ij}\,(t,x)\,dx^i dx^j \tag{1.3.13}$$

in the sense that

$$g_{\alpha\beta} = \Omega\left(t, x^i\right)\bar{g}_{\alpha\beta}\,. \tag{1.3.14}$$

Line elements of the form (1.3.13) were considered by Eisenhart in [20], while Lichnerowicz [22] introduced the line element (1.3.12), but none of them provided an explanation for their choice of metrics or a reason why the null dimensional reduction precisely works for this large class of metrics. The chain of arguments presented in this subsection is intended as a plausible line of reasoning leading to this choice.

Remark 1: Given an effective potential $\bar{V}$, eq.(1.3.11) shows that to any choice of time unit Ω corresponds distinct ambient metrics (1.3.12). Therefore, to a given Lagrangian system corresponds an infinite class of relativistic spacetimes not considered in [20].

Remark 2: Let us remind the reader that two Lagrangians L and L' are said to be equivalent if the actions differ by a total derivative, $L' = L + \frac{df}{dt}$, since their Euler-Lagrange equations are identical. In terms of the potentials, this is equivalent to a gauge transformation $\bar{A}_i' = \bar{A}_i + \partial_i f$ and $\bar{V}' = \bar{V} - \partial_t f$. From the point of view of the action, this means they differ by a boundary term, essentially equal to the variation of the function f. The interpretation of the variation of u as linear in the variation of the action along the trajectory suggests that the previous equivalence corresponds to the reparameterisations $u' = u + f(t,x)$. One can indeed check that the form (1.3.12) of the line element is preserved by this coordinate transformation, up to a gauge transformation of the potentials.

1.4 Ambient Lagrangian

Consider now the action principle $S\left[x^\mu\right] = \int \mathcal{L}\, d\tau$ for the geodesics parameterised by the affine parameter τ, on the ambient spacetime with line element (1.3.12), where the quadratic Lagrangian reads

$$\mathcal{L}\left[x^\mu, \frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\, g_{\alpha\beta}\left(t,x\right) \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{1.4.15}$$

The affine parameter τ is defined by the affine parameterisation constraint $\mathcal{L} = -\frac{M^2}{2}$, which is nothing but (1.3.10). The equations of motion read

$$\text{for } u: \quad \frac{d}{d\tau}\left(\Omega \frac{dt}{d\tau}\right) = 0 \tag{1.4.16}$$

$$\begin{aligned}\text{for } t: \quad & \frac{d}{d\tau}\left[\Omega\left(\frac{du}{d\tau} - 2\bar{U}\frac{dt}{d\tau} + \bar{A}_i \frac{dx^i}{d\tau}\right)\right] = -\frac{M^2}{2\Omega}\partial_t \Omega \\ & + \Omega\left(-\partial_t \bar{U}\left(\frac{dt}{d\tau}\right)^2 + \partial_t \bar{A}_i \frac{dt}{d\tau}\frac{dx^i}{d\tau} + \frac{1}{2}\partial_t \bar{g}_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right)\end{aligned} \tag{1.4.17}$$

$$\begin{aligned}\text{for } x^i: \quad & \frac{d}{d\tau}\left[\Omega\left(\bar{g}_{ij}\frac{dx^j}{d\tau} + \bar{A}_i \frac{dt}{d\tau}\right)\right] = -\frac{M^2}{2\Omega}\partial_i \Omega \\ & + \Omega\left(-\partial_i \bar{U}\left(\frac{dt}{d\tau}\right)^2 + \partial_i \bar{A}_j \frac{dt}{d\tau}\frac{dx^j}{d\tau} + \frac{1}{2}\partial_i \bar{g}_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}\right)\end{aligned} \tag{1.4.18}$$

where the affine parameterisation constraint $\mathcal{L} = -\frac{M^2}{2}$ has been used to simplify (1.4.17)-(1.4.18). We can solve eq.(1.4.16) in the form of (1.3.7) where m is now interpreted as a constant of motion, $\frac{dm}{d\tau} = 0$. This conservation law comes from the fact that the Lagrangian (1.4.15) does not depend on u. Thus the condition (1.3.7) is obtained as an equation of motion. Two cases must be distinguished: $m = 0$ and $m \neq 0$. The particular case $m = 0$ corresponds to the geodesics that entirely belong to a given hypersurface t=const since $dt/d\tau = 0$. Contrary to the generic case $m \neq 0$, these curves have no interpretation as dynamical trajectories: they are the rays of the congruence.

• $m = 0$, $M^2 = 0$ (null rays): If the geodesic is lightlike then the affine parameterisation constraint (1.3.10) with $dt = 0$ implies that $dx^i/d\tau = 0$. The latter equation together with $dt/d\tau = 0$ inserted into the equation of motion (1.4.17) imply that $du/d\tau$=const, since $\Omega(t,x)$=const. In conclusion, the lightlike geodesics belonging to a hypersurface of constant t are curves with x^i constant and with u as an affine parameter. These are the graviton worldlines defining the gravitational wave. As one can see, they generate the hypersurfaces t=const which are called "wavefront worldvolumes". A locus $u = f(t,x)$ defines a screen of detection/emission.

• $m = 0$, $M^2 < 0$ (spatial trajectories): One can check that the spacelike geodesics are at the same time geodesics $x^\mu(\tau)$ of the D-dimensional ambient spacetime and project onto spatial geodesics $x^i(\tau)$ of the metric $g_{ij} = \Omega\, \bar{g}_{ij}$. This can be seen by checking that eq.(1.4.18) with $dt/d\tau = 0$ is equivalent to the geodesic equation for the metric g_{ij} and the affine parameterisation constraint reads $\mathcal{L} = \frac{1}{2}\, g_{ij} \frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = -\frac{M^2}{2}$. In this sense, the wavefront worldvolumes t =const are totally geodesic submanifolds of the ambient spacetime.

• $m \neq 0$ (dynamical trajectories): In the generic case $m \neq 0$, one can reexpress eqs(1.4.17)-(1.4.18) as:

$$\ddot{u} - \partial_t \bar{U} - 2\partial_i \bar{U} \dot{x}^i + \partial_i \bar{A}_j \dot{x}^i \dot{x}^j + \bar{A}_i \ddot{x}^i - \frac{1}{2}\partial_t \bar{g}_{ij}\dot{x}^i\dot{x}^j + \frac{M^2}{2m^2}\partial_t\Omega = 0 \quad (1.4.19)$$

$$\ddot{x}^m + \bar{\Gamma}^m_{lj}\dot{x}^l\dot{x}^j + \bar{g}^{km}\left[\dot{x}^i\left(\partial_t\bar{g}_{ki} + \partial_i\bar{A}_k - \partial_k\bar{A}_i\right) + \partial_k\bar{U} + \partial_t\bar{A}_k\right] + \frac{M^2}{2m^2}\partial_k\Omega\bar{g}^{km} = 0 \quad (1.4.20)$$

We can put eq.(1.4.20) in the form of the Euler-Lagrange equation for the original Lagrangian (1.3.1)

$$\ddot{x}^m + \bar{\Gamma}^m_{lj}\dot{x}^l\dot{x}^j + \bar{g}^{km}\left[\left(\partial_t\bar{g}_{ki} + \bar{F}_{ik}\right)\dot{x}^i - \bar{E}_k\right] = 0 \quad (1.4.21)$$

where we introduced the spatial Levi-Civita connection $\bar{\Gamma}^m_{lj}$, the magnetic field strength

$\bar{F}_{ik} = \partial_i \bar{A}_k - \partial_k \bar{A}_i$ and the electric field $\bar{E}_k = -\partial_k \bar{V} - \partial_t \bar{A}_k$ together with the definition (1.3.11). Moreover, it can be checked that eq.(1.4.19) is compatible with the expression for $\dot{u}$ coming from the affine parameterisation constraint (1.3.10).

This completes the explicit check that the geodesics with $m \neq 0$ for the ambient spacetime (1.3.12) correspond to dynamical trajectories for the Lagrangian (1.3.1) in terms of the coordinates x^i and t so that the Eisenhart-Lichnerowicz theorem can now be formulated as:

Theorem 1.4.1 (Eisenhart-Lichnerowicz)**.** *The null dimensional reduction along the direction u of the affine geodesic equation for a curve $x^\mu(\tau) = \big(u(\tau), t(\tau), x^i(\tau)\big)$ parameterised by the affine parameter τ, satisfying $\frac{dt}{d\tau} \neq 0$ and $g_{\mu\nu}\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau} = -M^2$ on a manifold endowed with the metric*

$$ds^2 = \Omega(t,x)\left[2\,dt\left(du + \bar{A}_i(t,x)\,dx^i - \bar{U}(t,x)\,dt\right) + \bar{g}_{ij}(t,x)\,dx^i dx^j\right]$$

reduces to the Euler-Lagrange equations of the holonomic dynamical system characterised by the Lagrangian

$$L(t,x,\dot{x}) = \frac{1}{2}\bar{g}_{ij}(t,x)\,\dot{x}^i\dot{x}^j + \bar{A}_i(t,x)\,\dot{x}^i - \bar{V}(t,x)$$

where the effective potential $\bar{V}$ reads $\bar{V} = \bar{U} + \frac{1}{2}\frac{M^2}{m^2}\Omega$, with $m = \Omega\frac{dt}{d\tau}$.

We remind the reader that the extra coordinate u can be interpreted as the value of the action evaluated along the trajectory.

1.5 Hamiltonian perspective

The momenta corresponding to the Lagrangian (1.3.1) are given by $p_i = \bar{g}_{ij}(t,x)\,\dot{x}^j + \bar{A}_i(t,x)$. Thus the Hamiltonian reads

$$H(t,x^i,p_j) = \frac{1}{2}\bar{g}^{ij}(t,x)\Big(p_i - \bar{A}_i(t,x)\Big)\Big(p_j - \bar{A}_j(t,x)\Big) + \bar{V}(t,x) \qquad (1.5.22)$$

where $\bar{g}^{ij}$ denotes the inverse of the metric $\bar{g}_{ij}$. Obviously, this Hamiltonian function is the most general polynomial of degree two in the momenta with a positive-definite quadratic form as leading term.

The connection between the Hamiltonian action principles for the dynamical trajectories and for the ambient geodesics will be manifest in the "parameterised" Hamiltonian formulation obtained from the Lagrangian formulation where $t(\tau)$ is taken as a dynamical degree of freedom. The detailed Hamiltonian analysis [4] of such a system leads to the following action principle:

$$S[t, x^i, p_t, p_j, \lambda] = \int \left[p_i \frac{dx^i}{d\tau} + p_t \frac{dt}{d\tau} - \lambda \Big(p_t + H\left(t, x^i, p_j\right) \Big) \right] d\tau \tag{1.5.23}$$

where p_t is the conjugate of the (now dynamical) variable t while λ is the Lagrange multiplier for the first-class [5] constraint $p_t + H = 0$ corresponding to the reparameterisation invariance of the parameter τ. Solving the constraint as $p_t = -H$ inside the action gives the equivalent action principle

$$S[t, x^i, p_j] = \int \left[p_i \frac{dx^i}{d\tau} - H\left(t, x^i, p_j\right) \frac{dt}{d\tau} \right] d\tau \tag{1.5.24}$$

where the reparameterisation invariance $\tau \to \tau' = \tau'(\tau)$ can be used to impose the gauge fixation $dt/d\tau = 1$ in order to get the usual action principle $S[x^i, p_j] = \int \left[p_i \dot{x}^i - H\left(t, x^i, p_j\right) \right] dt$.

Now let us consider the parameterised Hamiltonian formulation of a free relativistic particle of mass M propagating on the ambient spacetime with line element (1.3.12) that arises from the Lagrangian $\mathcal{L}' = -M\sqrt{\left|g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right|}$:

$$\mathcal{S}[x^\mu, p_\nu, \lambda] = \int \left[p_\mu \frac{dx^\mu}{d\tau} - \frac{\lambda}{2}\, \Omega\left(p^2 + M^2\right) \right] d\tau\,, \tag{1.5.25}$$

with $p_u = \Omega \frac{dt}{d\tau}$ and λ a Lagrange multiplier for the mass-shell constraint $p^2 + M^2 = 0$ and where

$$\begin{aligned} p^2 &= g^{\mu\nu} p_\mu p_\nu \\ &= \Omega^{-1}(t,x) \Big[2\, p_t p_u + \bar{g}^{ij}(t,x) \Big(p_i - \bar{A}_i(t,x)\, p_u \Big) \Big(p_j - \bar{A}_j(t,x)\, p_u \Big) \\ &\qquad\qquad + 2\, \bar{U}(t,x)\, p_u^2 \Big]\,. \end{aligned} \tag{1.5.26}$$

As one can see, the form of the inverse metric $g^{\mu\nu}$ can be characterised as the most general ambient inverse metric that is independent of u and such that $g^{t\mu} \propto \delta^{u\mu}$. These two properties turn out to be the only two crucial ingredients in the null dimensional reduction of the Hamiltonian. This again provides a justification for the line element (1.3.12).

4. See *e.g.* [87] for more details on parameterised systems and their Hamiltonian constraints. Let us stress that, in the parameterised Hamiltonian formulation, the canonical Hamiltonian vanishes because of the time reparameterisation invariance.

5. A single constraint is automatically first class.

When $p_u \neq 0$, it turns out to be convenient to define

$$\bar{U} = \bar{V} - \frac{1}{2}\frac{M^2}{p_u^2}\,\Omega\,, \tag{1.5.27}$$

because inserting (1.5.26)-(1.5.27) inside (1.5.25) leads to a form of the action which is suggestively close to (1.5.23):

$$\mathcal{S}[x^\mu, p_\nu, \lambda] = \int d\tau \left[p_i \frac{dx^i}{d\tau} + p_t \frac{dt}{d\tau} + p_u \frac{du}{d\tau} - \lambda\left(p_t p_u + \mathcal{H}(t, x^i, p_j, p_u)\right)\right]\,, \tag{1.5.28}$$

with

$$\begin{aligned}\mathcal{H}(t, x^i, p_j, p_u) \quad &= \quad \frac{1}{2}\,\bar{g}^{ij}\,(t,x)\left(p_i - \bar{A}_i\,(t,x)\,p_u\right)\left(p_j - \bar{A}_j\,(t,x)\,p_u\right)\\ &\qquad + \bar{V}\,(t,x)\,p_u^2\,.\end{aligned} \tag{1.5.29}$$

The form of this Hamiltonian is the most general function of x^μ and p_μ that is a homogeneous polynomial of degree two in the momenta *and* independent of u and p_t. It can be seen as the homogenisation of the original Hamiltonian (1.5.22).

The main difference between the ambient action principle (1.5.28)-(1.5.29) and the reduced action principle (1.5.22)-(1.5.23) is the dependence on the conjugate pair of variable u and p_u. The decisive observation is that, since there is no explicit dependence on the variable u in the Hamiltonian (1.5.29), the conjugate momentum $p_u = \Omega \frac{dt}{d\tau} = m$ is a constant of motion. Therefore, it will not play any role in the Hamilton equations for the remaining variables which will thus be essentially the same as the original system. This proves the Eisenhart-Lichnerowicz Theorem without the need for performing any tedious computation. In Hamiltonian language, this Theorem may be phrased simply as follows: the original system (1.5.22)-(1.5.23) can be seen as the symplectic reduction of the system (1.5.28)-(1.5.29) through the addition of the extra constraint $p_u - m = 0$, which is first-class since $\mathcal{H}$ is independent of u. In other words, the action principle (1.5.23) is equivalent to the action principle

$$\begin{aligned}S[x^\mu, p_\nu, \lambda, \mu] \quad &= \quad \int d\tau \Big[p_i \frac{dx^i}{d\tau} + p_t \frac{dt}{d\tau} + p_u \frac{du}{d\tau}\\ &\qquad - \lambda\left(p_t p_u + \mathcal{H}(t, x^i, p_j, p_u)\right) - \mu\big(p_u - m\big)\Big]\,,\end{aligned} \tag{1.5.30}$$

where μ is a new Lagrange multiplier enforcing the constraint $p_u = m$.

Retrospectively, from the parameterised Hamiltonian perspective the main trick behind the ambient approach to dynamical trajectories is the homogenisation of the constraint $p_t + H(t, x^i, p_j) = 0$ to get a constraint $p_t p_u + \mathcal{H}(t, x^i, p_j, p_u) = 0$ that is quadratic in the momenta, via the introduction of an auxiliary momentum coordinate. The resulting constraint is a nondegenerate quadratic polynomial in the momentum with Lorentzian signature and can therefore be interpreted as the mass-shell constraint $p^2 + M^2 = 0$ of a free relativistic particle. There is an arbitrariness in such an identification which is reflected in the relation (1.5.27).

As a side remark, one may notice that by dividing (1.5.29) by p_u, one may see that the auxiliary momentum p_u actually plays the role of a nonrelativistic mass (*e.g.* the kinetic term of the "light-cone Hamiltonian" $\mathcal{H}/p_u$ is of the form $\vec{p}^2/2m$). This remark provides a nice interpretation of the action obtained from (1.5.28) after solving the mass-shell constraint as $p_t = -\mathcal{H}/p_u$ and fixing the reparameterisation invariance by $\tau = t$:

$$S[x^i, u, p_j, p_u] = \int \left[p_i \dot{x}^i + p_u \dot{u} - \frac{\mathcal{H}\left(t, x^i, p_j, p_u\right)}{p_u} \right] dt. \tag{1.5.31}$$

This interpretation of the auxiliary momentum p_u as a nonrelativistic mass is standard when the ambient spacetime is Minkowski (or AdS) spacetime. In such cases, the ambient approach essentially coincides with the light-cone formalism, but a remarkable fact is that this setting actually generalises smoothly to the much wider class of curved spacetimes with line element (1.3.12) that will be motivated and described more geometrically in the following.

1.6 Gravitational waves and Plato's allegory

In order to understand better the heuristics behind the ambient approach, let us describe the former spacetimes in more geometric terms, starting to sketch some technical details and motivating our future choices of terminology.

Consider the propagation of a gravitational wave in the ambient spacetime and a screen detecting the gravitons passing by. In a spacetime diagram, the worldlines of gravitons are null rays, *i.e.* they define a null geodesic congruence, and the registered events on the screen are simply intersections between the screen worldvolume and the null rays. So, technically, the screen worldvolume is a codimension-one hypersurface which is transverse to the congruence of null rays, in the sense that each ray intersects it only once (Fig. 1.1). The events are encoded via the position on the screen and the instant of the intersection. Heuristically, these coordinates on the screen worldvolume already suggest that the former might be endowed with a natural structure of (codimension-one) spacetime. In order to push the spacetime picture further, consider the screen at any given instant as a wavefront. From a spacetime point of view, the propagation of this wavefront translates into the fact that null rays generate the corresponding wavefront worldvolume, each such hypersurface

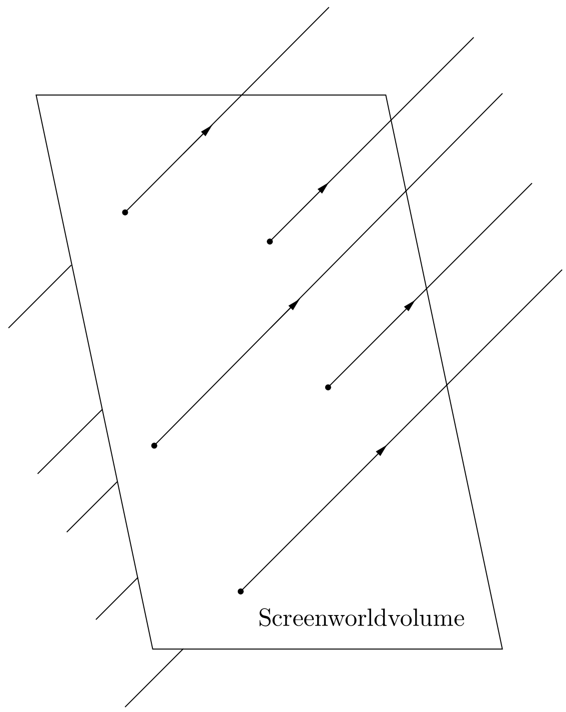


Figure 1.1: The screen worldvolume is transverse to the congruence of null rays. (In all figures, we will follow the standard spacetime diagram convention, *i.e.* time flows from bottom to top and null directions are at 45°.)

is labeled by the time of emission, the "retarded" time (Fig. 1.2). The family of these wavefront worldvolumes provides a foliation of the ambient spacetime the leaves of which are orthogonal to the null rays. Retrospectively, this provides a geometric definition for a gravitational wave as a foliated spacetime. The screen worldvolume can then be thought as a codimension-one hypersurface transverse to this foliation, such that the intersection between a leave and the screen worldvolume is precisely the instantaneous screen we started with.

The projection on the screen along rays maps the ambient spacetime on a codimension-one manifold endowed with a notion of time induced from the foliation of the ambient spacetime: the retarded time. If the relativistic structure (*i.e.* the metric) of the ambient spacetime is preserved along the rays (*i.e.* they are Killing orbits), then it can induce a well-defined structure on the quotient space which can be represented as a screen worldvolume. The remarkable fact is that this projection defines a nonrelativistic spacetime structure (*i.e.* absolute rulers and clocks) on the screen worldvolume.[6]

6. By construction, this structure does not depend on the specific choice of screen worldvolume, for instance two screens in relative motions would encode the same geometric data with respect to their rulers and clocks.

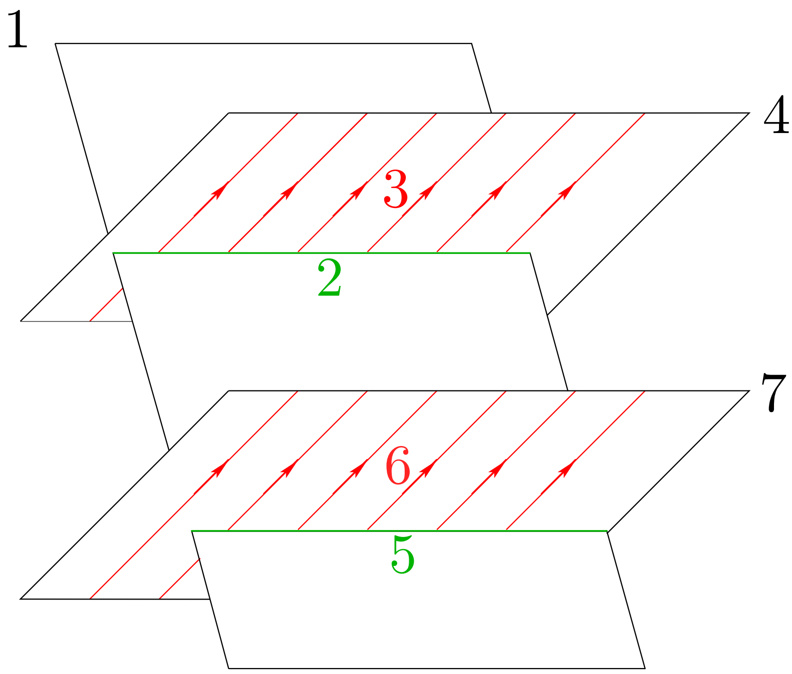


Figure 1.2: 1. Screen worldvolume; 2. Screen at $t = t_1$; 3. Congruence of null geodesics generating the wavefront worldvolume $t = t_1$; 4. Wavefont worldvolume $t = t_1$; 5. Screen at $t = t_0$; 6. Congruence of null geodesics generating the wavefront worldvolume $t = t_0$; 7. Wavefont worldvolume $t = t_0$

Actually, the induced line element on the screen worldvolume encodes more information than absolute clocks and rulers but is equivalent to the specification of a Lagrangian for a holonomic dynamical system. Perhaps even more remarkable is that the projections of ambient geodesics on the screen have a natural interpretation as dynamical trajectories of nonrelativistic particles (Fig. 1.3). Furthermore, between the emission of a graviton by the geodesic and its detection on the screen, the affine parameter along the null ray is equal to the value of the action (modulo two fixed constants: a multiplicative and an additive one). In other words, if the physicist knows the shadows of all geodesics together with the value of this affine parameter, then she/he is able to reconstruct the ambient spacetime. This procedure provides a concrete description of the Eisenhart lift. In a sense one might say that if the value of the action is considered as a sort of extra coordinate that one should add to the absolute space and time coordinates for the description of nonrelativistic dynamical trajectories, then the corresponding constructed spacetime with one more dimension admits a natural description in terms of a gravitational wave.

Spacetimes with a null hypersurface-orthogonal Killing vector field have already been investigated in the literature [72, 88] but, to our knowledge, no specific name has been given to this wide class of spacetimes. Since this is the one relevant for the ambient approach

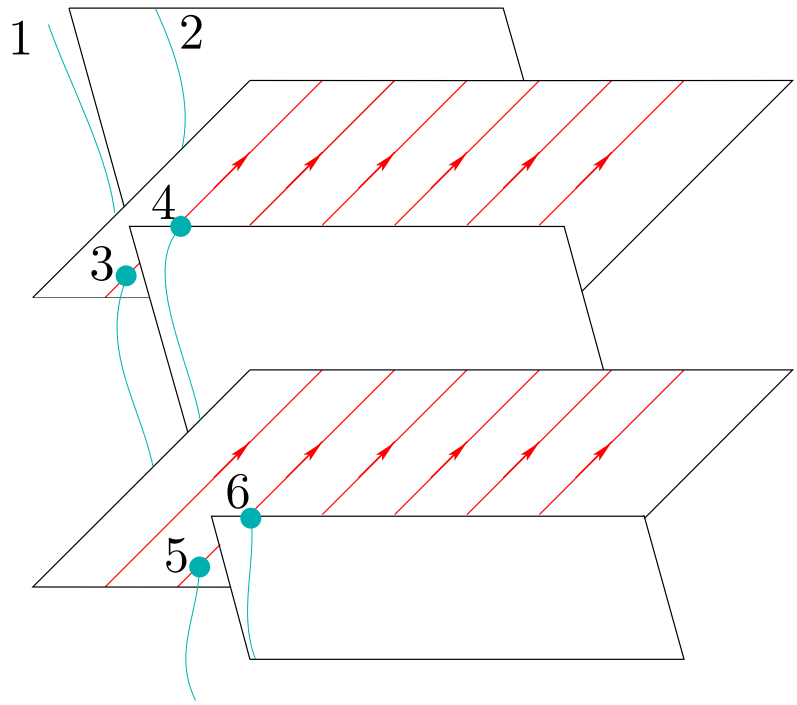


Figure 1.3: The Eisenhart lift, 1. Geodesic of ambient spacetime; 2. Shadow of the ambient geodesic on the screen worldvolume; 3. Emission of a graviton by the geodesic; 4. Detection on the screen at $t = t_1$; 5. Emission of a graviton by the geodesic; 6. Detection on the screen at $t = t_0$;

and as a tribute for the stimulating analogy [88] with the allegory of the cave, we will refer to such a spacetime as a "Platonic gravitational wave". Accordingly, its orbit space of null rays will be called "Platonic screen". The projection of ambient objects (such as clocks, geodesics, *etc.*) on this screen will be called their "shadows".

1.7 Schrödinger equation from Klein-Gordon equation

As shown in Section (1.5), the Eisenhart-Lichnerowitz Theorem for the classical particle acquires a simpler formulation when seen from a Hamiltonian perspective. In the present Section, the Theorem is extended to first-quantised equations for a scalar field, *i.e.* the Schrödinger equation is derived as a null dimensional reduction of the Klein-Gordon equation. In a first step, we review the results of [24, 33] by performing the reduction for the line element (1.3.13) before generalising these results to the conformally equivalent class (1.3.12).

According to the standard rules of quantisation, the momenta appearing in the classical Hamiltonian formalism are essentially converted into partial derivatives and the Hamiltonian turns into an operator such that the mass-shell constraint becomes the Klein-Gordon equation. One then faces the ambiguity because of the introduction of noncommuting operators. We choose to fix the ambiguity by focusing on the conformal invariant Laplacian of Yamabe, in order to take advantage of the conformal relation between the classes of spacetimes at hand. This formalism is reminiscent of the light-cone formulation [89] and can be seen as a generalisation thereof to suitable curved spacetimes.

Starting with the D-dimensional Klein-Gordon action:

$$S = \int d^D x \sqrt{-g}\, \Phi^* \left(\Box_Y \Phi - M^2 \Phi\right), \tag{1.7.32}$$

whose equations of motion read

$$\Box_Y \Phi - M^2 \Phi = 0\,, \tag{1.7.33}$$

where $\Box_Y = \Box - \frac{D-2}{4(D-1)} R$ is the Yamabe operator, with $\Box \equiv \nabla^\mu \nabla_\mu$ the Laplace-Beltrami operator. The Yamabe operator is also known as the conformal Laplacian, because of the conformal invariance of the equation $\Box_Y \Phi = 0$ (*cf.* *e.g.* appendix D of [90]). More precisely, if g and $\bar{g}$ are conformally related via $g = \Omega \bar{g}$, then the equation $\Box_Y \Phi = 0$ is said to be conformally invariant with weight $-\frac{d}{4}$ (where $d = D - 2$), *i.e.* it satisfies:

$$\Box_Y \left(\Omega^{-\frac{d}{4}} \Phi\right) = \Omega^{-1-\frac{d}{4}} \bar{\Box}_Y \Phi. \tag{1.7.34}$$

We start by considering the line element (1.3.13) (this class of metrics will be referred to as Bargmann-Eisenhart waves in the following Sections) and perform the dimensional reduction of the action (1.7.32) along the lightlike direction $\frac{\partial}{\partial u}$ by considering a specific Fourier mode in the direction u: $\Phi(u, t, \vec{x}) = \phi(t, \vec{x}) e^{imu}$ (*cf.* [24, 33]). As can be easily checked, the scalar curvature and determinant of the metric (1.3.13) are equal to the ones of the spatial metric $\bar{g}_{ij}$ so we have $\bar{R} = \bar{R}^{(d)}$ and $\det \bar{g} = \det \bar{g}^{(d)}$.

The action (1.7.32) then reduces to:

$$\begin{aligned} S \;=\; \int d^D x \sqrt{\bar{g}^{(d)}}\, \phi^* \Bigg[& D^2 \phi + 2im\partial_t \phi + \frac{1}{2} im \partial_t \left(\ln \bar{g}^{(d)}\right) \phi \\ & - \left(M^2 + 2m^2 \bar{U} + \frac{d}{4\,(d+1)} \bar{R}^{(d)}\right) \phi \Bigg] \end{aligned} \tag{1.7.35}$$

where we introduced the covariant derivative $D_i \phi = \bar{\nabla}_i^{(d)} - im\bar{A}_i$. For cosmetic reasons, the term involving the time derivative of the determinant for the metric $\bar{g}$ can be integrated by parts to obtain:

$$S = \int d^D x \sqrt{\bar{g}^{(d)}} \left(\phi^* D^2 \phi + 2m^2 \rho - \left(M^2 + 2m^2 \bar{U} + \frac{d}{4\,(d+1)} \bar{R}^{(d)}\right) |\phi|^2\right) \quad (1.7.36)$$

where ρ stands for the density probability: $\rho = \frac{i}{2m}\left(\phi^* \partial_t \phi - \phi \partial_t \phi^*\right)$. The associated equations of motion then read:

$$\begin{aligned} \bar{\Box}_Y \Phi - M^2 \Phi \;&=\; e^{imu} \Big[D^2 \phi - 2m^2 \bar{U} \phi + 2im\partial_t \phi \\ &\quad + \frac{1}{2} im \partial_t \left(\ln \bar{g}^{(d)}\right) \phi - \frac{d}{4\,(d+1)} \bar{R}^{(d)} \phi - M^2 \phi \Big] = 0 \end{aligned} \quad (1.7.37)$$

so that Klein-Gordon equation on the curved spacetime (1.3.13) reduces to Schrödinger equation on the curved space $\bar{g}_{ij}$ (*cf.* *e.g.* [91]):

$$i\partial_t \phi = \left[-\frac{1}{2m} \left(D^2 + \frac{d}{4\,(d+1)} \bar{R}^{(d)} \right) + m\, \bar{V}' - \frac{i}{4} \partial_t \left(\ln \bar{g}^{(d)} \right) \right] \phi \quad (1.7.38)$$

where we defined $\bar{V}' = \bar{U} + \frac{M^2}{2m^2}$. The operator $i\partial_t + \frac{1}{2m}\left(D^2 + \frac{d}{4(d+1)} \bar{R}^{(d)}\right) + \frac{i}{4} \partial_t \left(\ln \bar{g}^{(d)}\right)$ can be seen as a nonrelativistic equivalent of the Yamabe operator.

We now switch to the class of metrics whose line element takes the form (1.3.12) (later referred to as Platonic waves), which are conformally related to the previously studied class as we have $g = \Omega\,(t, x)\,\bar{g}$. The choice of the Yamabe operator then turns out to be handy, thanks to the property (1.7.34) which suggests the following ansatz: $\Phi(u, t, \vec{x}) = \Omega^{-d/4} \phi(t, \vec{x}) e^{imu}$ under which the action (1.7.32) becomes:

$$\begin{aligned} S = \int d^D x \sqrt{\bar{g}^{(d)}}\, \phi^* \Big[& D^2 \phi + 2im\partial_t \phi + \frac{1}{2} im \partial_t \left(\ln \bar{g}^{(d)}\right) \phi \\ & - \left(M^2 \Omega + 2m^2 \bar{U} + \frac{d}{4\,(d+1)} \bar{R}^{(d)} \right) \phi \Big]. \end{aligned} \quad (1.7.39)$$

The associated equations of motion read

$$\begin{aligned} & \Box_Y \Phi - M^2 \Phi = \Omega^{-1-\frac{d}{4}} e^{imu} \Big[D^2 \phi + 2im\partial_t \phi \\ & + \frac{1}{2} im \partial_t \left(\ln \bar{g}^{(d)}\right) \phi - \left(M^2 \Omega + 2m^2 \bar{U} + \frac{d}{4\,(d+1)} \bar{R}^{(d)} \right) \phi \Big] = 0 \end{aligned} \quad (1.7.40)$$

which once again leads to Schrödinger equation:

$$i\partial_t\phi = \left[-\frac{1}{2m}\left(D^2 + \frac{d}{4\,(d+1)}\bar{R}^{(d)}\right) + m\,\bar{V} - \frac{i}{4}\partial_t\left(\ln\bar{g}^{(d)}\right)\right]\phi \tag{1.7.41}$$

with $\bar{V} = \bar{U} + \frac{M^2\Omega}{2m^2}$.

Chapter 2

Platonic waves

Similarly to the definition of manifolds endowed with a *Riemannian* structure, *i.e.* a positive-definite metric, one can define *relativistic spacetimes* as smooth manifolds endowed with a *Lorentzian* structure, *i.e.* a metric with signature $(-,+,...,+)$. Somewhat less familiar to most physicists are the *nonrelativistic spacetimes* which are smooth manifolds endowed with absolute clock and rulers or even absolute time and space (to be defined below). As will be shown, gravitational waves may hide such nonrelativistic structures inside their space of rays.

The notions of a gravitational wave (defined geometrically as a spacetime with a null hypersurface-orthogonal vector field), of a Bargmann-Eisenhart gravitational wave (= with parallel wave vector field) and of a Platonic gravitational wave (= conformal to a Bargmann-Eisenhart wave and with Killing wave vector field) are introduced together with the canonical form of their metric.

2.1 Embedding nonrelativistic structures

The present work deals with nonrelativistic features embedded inside relativistic spacetimes. In this context, one can legitimately ask: what constitutes the most general class of relativistic spacetimes inducing a nonrelativistic structure? In order to address this question, one needs first to properly define nonrelativistic structures. We will at first follow the definition of [92] of a Leibnizian structure, which will turn out to be too weak a requirement and next switch to the more restrictive notion of Aristotelian structure.
A *Leibnizian structure* [92] comprises the following three elements: a manifold $\mathcal{M}$, a one-form ψ and a positive-definite metric γ acting on the kernel of ψ (Everywhere in this work are vector fields and one-forms assumed to be nowhere vanishing. This assumption will often be left implicit for the sake of brevity. Similarly, manifolds are taken to be smooth

and connected.). We will call ψ an *absolute clock* and γ a collection of *rulers*. As such, it is easy to see that any relativistic spacetime induces a Leibnizian structure. Indeed, the tangent space to a D-dimensional relativistic spacetime is isomorphic to Minkowski spacetime and can be endowed at each point with a set of D orthogonal coframes $(e_0, e_1, ..., e_{D-1})$. Choosing $\psi \equiv e_0$ as an absolute clock, each point is endowed with a positive-definite metric acting on the kernel of ψ engendered by the vectors dual to the forms $e_1, ..., e_{D-1}$.

As is now manifest, the above definition of a nonrelativistic structure is too weak to discriminate a subclass of relativistic spacetimes. Furthermore, it does not allow a global definition of absolute time and space since it only provides a set of local clocks and rulers. These two drawbacks of the previous definition can be circumvented by the introduction of an extra condition on the one-form ψ. The requirement that the nonrelativistic structure allows a global notion of absolute time and space amounts to define submanifolds of $\mathcal{M}$ endowed with the spatial metric γ, *i.e.* they have to admit the kernel of the one-form ψ as tangent vector space. The necessary and sufficient condition for the existence of such integral submanifolds (*cf.* *e.g.* appendix B.3 of [90]) is the Frobenius integrability condition $\psi \wedge d\psi = 0$, so that the kernel of ψ defines a foliation of $\mathcal{M}$ by a family of hypersurfaces of codimension-one called *simultaneity slices*. These are the integral submanifolds endowed with the spatial metric γ. Locally, $\psi = \Omega\, dt$ where $\Omega > 0$ and the function t is called an *absolute time*. The simultaneity slices are the hypersurfaces of fixed absolute time and are called *absolute spaces*, as they are endowed with the positive-definite metric γ. We will call a Leibnizian structure whose absolute clock satisfies the Frobenius integrability condition an *Aristotelian* structure. They were called Leibnizian structures with *locally synchronisable* clock in [92]. [1]

In order to determine the class of relativistic spacetimes inducing an Aristotelian structure, we seek for spacetimes admitting a hypersurface-orthogonal vector field [the dual to the absolute clock ψ, denoted $\xi \equiv g^{-1}(\psi)$] and restrict for simplicity our analysis to the case where ξ is of definite type throughout the entire spacetime. We further restrain to cases when the transverse metric on the simultaneity slices is positive semidefinite, as seems natural in order to induce an Aristotelian structure on them (or a quotient thereof). As spacetimes admitting a spacelike hypersurface-orthogonal vector field necessarily induce a Lorentzian transverse metric, they do not constitute natural candidates in order to yield a positive-definite spatial metric. Therefore, we are left with the following two cases:

- $g(\xi, \xi) < 0$: Relativistic spacetimes admitting a timelike hypersurface-orthogonal

1. Note that the reference to Leibniz can seem somewhat improper since he actually debated with Newton and strongly argued against absolute time and space. We thus prefer to refer to Aristotle because Aristotelian physics is pre-relativist (even in the Galilean sense) and also does not include the inertial principle. Accordingly, our definition of Aristotelian structure does not involve any notion of parallelism (contrarily to a Galilean manifold, *cf.* [92, 24]).

vector field indeed induce an Aristotelian structure as the transverse metric to the vector field on the simultaneity slices is positive definite. This class of time-foliated spacetimes includes the Friedmann-Lemaître-Robertson-Walker spacetimes whose cosmological time labels the different slices which are homogeneous spaces. A peculiarity of time-foliated spacetimes is that they possess both relativistic and nonrelativistic features, *i.e.* the nonrelativistic spacetime merges with the relativistic spacetime, and not with a quotient thereof. This interesting class will not be considered further here, being already well studied and moreover stepping outside the scope of the present work which focuses on dimensional reduction.

• $g(\xi, \xi) = 0$: The lightlike case will constitute the main object of study of the present Section and associated relativistic spacetimes will be called *gravitational waves.*

2.1.1 Gravitational waves

The class of spacetimes with a null hypersurface-orthogonal vector field has the nice feature of allowing the introduction of a special chart of coordinates, the so-called *Brinkmann coordinates* [2] which induce a canonical form for the metric. This is actually the chart we used in Section 1 and which we will use extensively in the following. These spacetimes are also interesting since, as suggested by their name, they possess the minimal structure allowing a fruitful usage of wave-related features for their characterisation.

We start with some definitions: a *wave vector field* is a hypersurface-orthogonal null and complete vector field, the orbits of which are called *rays.*

Definition 2.1.1. *A gravitational wave is a Lorentzian structure possessing a wave vector field.*

Note that our definition of a gravitational wave is purely geometrical *i.e.* does not assume the metric to be solution of field equations. This is in contradistinction with the more standard use of the term among relativists where gravitational waves are solutions of vacuum Einstein equations.

The congruence of rays defines the gravitational wave via the standard rules of geometric optics. For instance, a *wavefront worldvolume* is a hypersurface which is orthogonal to the congruence of rays. Wavefront worldvolumes are thus codimension-one null hypersurfaces containing a (sub)congruence of rays (because the wave vector field is orthogonal to itself), *cf.* Fig.1.2. By definition, a gravitational wave is a spacetime foliated by the wavefront worldvolumes.

2. The term Brinkmann coordinates seems standard for pp-waves [93] but they were originally introduced for Bargmann-Eisenhart spacetimes [21]. Here we slightly generalise the denotation of this term.

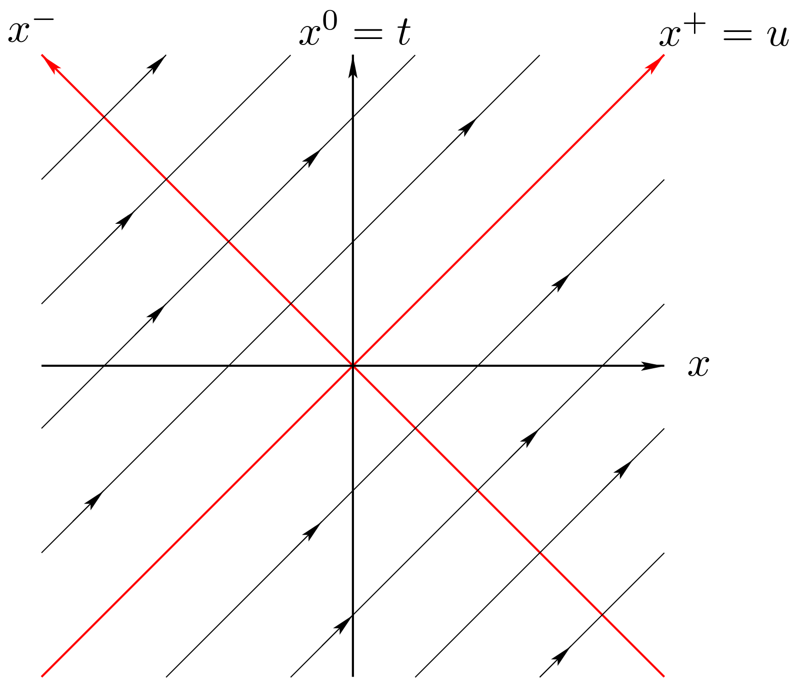


Figure 2.1: Two-dimensional Minkowski spacetime as a gravitational wave. The wavefront worldvolumes are the lines $x^- = \text{const}$.

Example: The simplest example of a gravitational wave (according to the above definition) is Minkowski spacetime. It can indeed be foliated by any collection of parallel null hyperplanes, interpreted as flat wavefront worldvolumes (Fig.2.1). The corresponding congruence of rays is provided by the parallel null lines inside each leave.

We will denote the wave vector field by ξ. The differential 1-form dual to ξ is referred to as the *wave covector field* and written $\psi \equiv g\,(\xi)$, the components of which are: $\psi_\mu \equiv g_{\mu\nu}\xi^\nu = \xi_\mu$. Due to the hypersurface-orthogonality condition on the wave vector field ξ, the wave covector field can be written locally as $\psi = \Omega\, df$ where the primitive f is called the *retarded time* (or "phase") and we assume without loss of generality that $\Omega > 0$. In components, this reads as $\xi_\mu = \Omega\,\partial_\mu f$. As one can see, the level sets of the retarded time (*i.e.* the loci $f = \text{constant}$) are the wavefront worldvolumes. Notice that, since the wave (co)vector field is null, $\mathcal{L}_\xi f = 0$ (since $0 = \xi_\mu \xi^\mu = \Omega\, \xi^\mu \partial_\mu f$).

Lemma 2.1.2. *The wave covector field defines a locally synchronisable absolute clock on a gravitational wave, whose absolute time is the retarded time and whose simultaneity slices are the wavefront worldvolumes.*

Example: Light-cone time x^- provides an absolute time on Minkowski spacetime (Fig. 2.2). Notice in this example, that contrary to nonrelativistic spacetimes, there may exist several inequivalent “absolute” times (for instance x^- or x^0 in fig.2.2) on relativistic spacetimes that admit inequivalent wave vector fields.

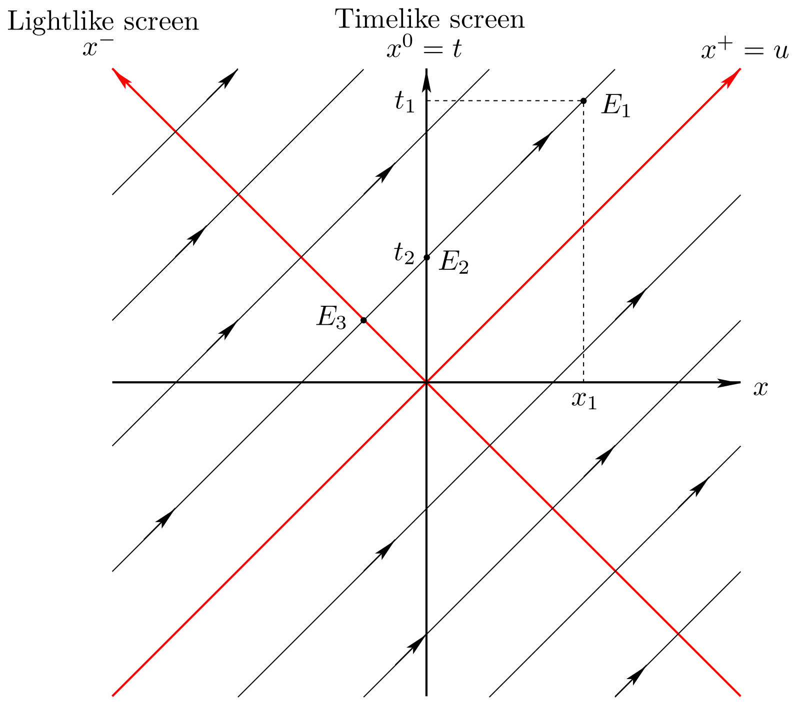


Figure 2.2: Several choices of screen worldvolumes are possible *e.g.* the timelike screen worldvolume axis x^0, so that leaves of the foliation are labeled by the retarded time t, or the lightlike screen worldvolume axis x^- which labels leaves with light-cone time x^-. The event E_1 is encoded on the timelike screen worldvolume by its position x_1 and the time of emission (E_2) of the graviton intersecting it: $t_2 = t_1 - x_1$. Alternatively, on the lightlike screen worldvolume, the moment of emission (E_3) of the graviton intersecting E_1 has for light-cone time $x^- = \frac{t_1 - x_1}{\sqrt{2}}$.

2.1.2 Brinkmann coordinates

The *Brinkmann coordinates* are now introduced as follows: two among the $D = n+1$ coordinate vector fields $\frac{\partial}{\partial x^\mu}$ are specialised, let us call them $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$. The first coordinate is taken to be the affine parameter u along rays (so the corresponding coordinate vector field is identified with the wave vector itself, $\frac{\partial}{\partial u} = \xi$); the second coordinate corresponds to the retarded time ($t = f$); and the remaining $d = n-1$ coordinates x^i are coordinate systems on the wavefronts.[3] Thus, one has $g_{u\mu} = g\left(\xi, \frac{\partial}{\partial x^\mu}\right) = \xi_\mu = \Omega\, \delta^t_\mu$. From this last relation, one sees that the remaining $d = n-1$ coordinate vector fields $\frac{\partial}{\partial x^i}$ are orthogonal

3. We follow the coordinate convention of [20] and [24] which differs from the standard notation in gravitational waves literature where our (u, t) coordinates are usually denoted (v, u) respectively.

to the null vector field, as they should since by construction the coordinates (u, x^i) must provide coordinates on the wavefront worldvolumes. Similarly, the coordinates (t, x^i) provide coordinates on the hypersurface $u = 0$ that can be interpreted as a screen worldvolume corresponding to the choice of transverse vector field $\frac{\partial}{\partial t}$.

In a Brinkmann coordinate chart, the line element thus takes the canonical form:

$$ds^2 = g_{tt}\, dt^2 \;+\; 2\,\Omega\, dt\, du \;+\; 2\, g_{ti}\, dx^i dt \;+\; g_{ij} dx^i dx^j\,,$$

where the metric components $g_{\mu\nu}$ are in general functions of all the coordinates. Looking backward to Section 1.3 or forward to Section 2.2, one can introduce the (scalar) potential $\bar{U} = -\frac{1}{2}\Omega^{-1} g_{tt}$, the Coriolis 1-form $\bar{A}_i = \Omega^{-1} g_{ti}$ and the conformally related spatial metric $\bar{g}_{ij} = \Omega^{-1} g_{ij}$ and reexpress the canonical line element as:

$$ds^2 = \Omega\,(t,x)\left[2\,dt\left(du + \bar{A}_i(u,t,x)\, dx^i - \bar{U}(u,t,x)dt\right) \;+\; \bar{g}_{ij}(u,t,x)\, dx^i dx^j\right] \qquad (2.1.1)$$

where, without loss of generality, Ω can be taken independent of u, as will be shown later. The inverse metric now reads:

$$g^{-1} = \Omega^{-1}\left[\left(2\bar{U} + \bar{g}^{ij}\bar{A}_i\bar{A}_j\right)\partial_u\otimes\partial_u + \partial_u\otimes\partial_t + \partial_t\otimes\partial_u - \bar{g}^{ij}\bar{A}_j\left(\partial_u\otimes\partial_i + \partial_i\otimes\partial_u\right) + \bar{g}^{ij}\partial_i\otimes\partial_j\right].$$

Example: The light-cone coordinates x^μ $(\mu = +, -, i)$, where $x^\pm = (x^0 \pm x^n)/\sqrt{2}$ on the Minkowski spacetime $\mathbb{R}^{n,1}$, provide Brinkmann coordinates for the simplest instance of a gravitational wave (Fig.2.2). The flat line element reads

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -2\, dx^+ dx^- + \delta_{ij} dx^i dx^j\,, \qquad (2.1.2)$$

so that one might identify the retarded time t with x^- and the affine parameter u with x^+.

It will be useful for some calculations to dispose of a frame version of the Brinkmann coordinates. A *light-cone frame* is a moving (co)frame where the line element takes the form

$$ds^2 = \eta_{ab} e^a e^b = -2\, e^+ e^- + \delta_{ij} e^i e^j. \qquad (2.1.3)$$

In the Petrov-type classifications, the vectors e^-, e^+, e^i are often denoted by ℓ, n, m^i, respectively [4]. An *adapted frame* is defined as a light-cone frame where the null frame $\ell \equiv e^-$ is taken to be the clock $\psi = g\,(\xi)$. The other null (co)frame $n \equiv -e^+$ is then

4. We keep this convention for the rest of this Section and adopt the following translation table for the

completely determined by the line element (2.1.3). Often the Brinkmann coordinates will be used, so that the null coframes will read $\ell = g\,(\xi) = \Omega\, dt$ and $n = du + \bar{A}_i\, dx^i - \bar{U} dt$.

There is no canonical prescription for the remaining "orthonormal" coframes $m^i \equiv e^i$ on the wavefronts, which must be such that

$$\delta_{ij} e^i e^j = g_{ij}\, dx^i dx^j\,.$$

As one can see from (2.1.3), e^+ and e^- being null, the (co)frames e^i must be spacelike in order for the spacetime metric $g_{\mu\nu}$ to have a Lorentzian signature, and so the metric g_{ij} must be positive definite.[5] However, the type of $\frac{\partial}{\partial t}$ (*i.e.* the sign of g_{tt} and $\bar{U}$) can be anything.

The 1-forms ℓ and n are also useful to covariantly define the *transverse metric*

$$^{\perp}\gamma_{\mu\nu} = g_{\mu\nu} - 2\, n_{(\mu}\ell_{\nu)} = g_{ij} e^i_{\mu} e^j_{\nu} \tag{2.1.5}$$

with $n^2 = \ell^2 = 0$ and $n \cdot \ell = 1$. It is easy to check that the wave vector field $\xi = \frac{\partial}{\partial u}$, as well as $\frac{\partial}{\partial t}$, belongs to the radical of $^{\perp}\gamma$. The transverse metric $^{\perp}\gamma$ is necessary in order to define the *optical scalars* associated to the wave vector field ξ, *i.e.* the expansion $\theta = \nabla^{\alpha}\xi_{\alpha}$, the shear σ and the twist ω. The transverse part of the tensor $\nabla\xi$ can indeed be decomposed into its $\mathfrak{o}(d)$-irreducible parts as $^{\perp}\gamma^{\alpha}_{\mu}{}^{\perp}\gamma^{\beta}_{\nu}\nabla_{\beta}\xi_{\alpha} = \frac{1}{d}\,\theta\,^{\perp}\gamma_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$ with $\sigma_{\mu\nu} = \sigma_{(\mu\nu)}$ and $^{\perp}\gamma^{\mu\nu}\sigma_{\mu\nu} = 0$ and $\omega_{\mu\nu} = {}^{\perp}\gamma^{\alpha}_{[\mu}{}^{\perp}\gamma^{\beta}_{\nu]}\nabla_{\beta}\xi_{\alpha} = {}^{\perp}\gamma^{\alpha}_{\mu}{}^{\perp}\gamma^{\beta}_{\nu}\nabla_{[\beta}\xi_{\alpha]}$. The shear σ and twist ω are the scalar fields respectively defined by $\sigma^2 = \frac{1}{2}\,\sigma^{\mu\nu}\sigma_{\mu\nu} = \frac{1}{2}\,\sigma^{ij}\sigma_{ij}$ and $\omega^2 = \frac{1}{2}\,\omega^{\mu\nu}\omega_{\mu\nu} = \frac{1}{2}\,\omega^{ij}\omega_{ij}$. Since σ^2 and ω^2 are sums of squares, the shear σ and the twist ω respectively vanish if and only the tensors $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ respectively vanish.

Remark: We stress that the "rotational" two-form (or "curl") $d\xi$ with components $\partial_{[\mu}\xi_{\nu]} = \nabla_{[\mu}\xi_{\nu]}$ and the "rotation" (or "twist") two-form ω with components $\omega_{\mu\nu} = {}^{\perp}\gamma^{\alpha}_{\mu}{}^{\perp}\gamma^{\beta}_{\nu}\nabla_{[\beta}\xi_{\alpha]}$ are in general distinct tensors. Indeed, they must be distinguished for null forms, although they coincide for time (or space) like ones. In fact, from Frobenius Theorem one knows that a wave vector field is automatically twistless, although it is not necessarily irrotational.

rest of the manuscript

$$\begin{cases} l = \psi \\ n = \overset{N}{A} \\ m = e^i \\ ^{\perp}\gamma = \overset{N}{\gamma}. \end{cases} \tag{2.1.4}$$

5. The positive-definiteness of the spatial metric γ is also obvious from the calculation of the determinant of the ambient metric (2.1.1) which reads $\det g = -\Omega^D \det \gamma$.

In the Brinkmann coordinates, the kernel of ψ at each point of the simultaneity slices is the n-dimensional vector space composed of tangent vectors X satisfying $g\left(X, \frac{\partial}{\partial t}\right) = 0$. Therefore, for X, Y belonging to the kernel of ψ, the action of g writes

$$g\,(X, Y) = g_{ij} X^i Y^j = \gamma\,(X, Y) = {}^{\perp}\gamma\,(X, Y)$$

so the induced (or transverse) metric ${}^{\perp}\gamma$ on the simultaneity slices is of rank $d = n - 1$ and its action reduces to the one of the positive-definite d-dimensional spatial metric γ. The wavefront worldvolumes are then endowed with a positive semidefinite metric ${}^{\perp}\gamma$ and then, as such, cannot be given the interpretation of absolute spaces. In order to obtain a nondegenerate metric, one can quotient the wavefront worldvolume by the null direction. However, this procedure is only well-defined if the rays are orbits of an isometry. As we will argue, this further requirement is necessary in order for a gravitational wave to induce an Aristotelian structure. The next subsection is devoted to a description of this quotient manifold.

2.1.3 Platonic screens

The previous "gravitational wave" terminology is further justified when one considers the following Lemma:

Lemma 2.1.3. *Any wave vector field is geodesic.*

Consequently, rays are null geodesics and can thus be interpreted as graviton worldlines.

Proof: Using the hypersurface-orthogonality of the vector field and Frobenius Theorem, we see that the 1-form $\psi \equiv g\,(\xi)$ satisfies $d\psi = \alpha \wedge \psi$ for some 1-form α. Expressing the left-side in terms of Koszul connections and contracting with ξ, one obtains $\nabla_\xi \xi - \frac{1}{2}\nabla\left(\xi^2\right) = (\alpha \cdot \xi)\,\xi - \left(\xi^2\right)\alpha$, which, for a null vector field ($\xi^2 = 0$), is equivalent to the geodesic condition [6]. $\square$

Proposition 2.1.4. *Any gravitational wave admits an affine geodesic wave vector field.*

Proof: Let ξ' be a wave vector field and ξ the vector field defined by $\xi' = f\xi$, with f a non-vanishing function. The vector field ξ is null since $g\,(\xi, \xi) = f^{-2} g\,(\xi', \xi') = 0$. Furthermore, denoting $\psi' \equiv g\,(\xi')$ and $\psi \equiv g\,(\xi)$, the hypersurface-orthogonality condition of ξ' reads $d\psi' \wedge \psi' = d\,(f\psi) \wedge (f\psi) = f df \wedge \psi \wedge \psi + f^2 d\psi \wedge \psi = f^2 d\psi \wedge$

6. A comment on the terminology is in order. In this work, the term *geodesic* will be used to designate not-necessarily affinely-parameterised geodesic vector fields (*i.e.* satisfying $\nabla_\xi \xi = \kappa\, \xi$ with κ a function of coordinates) and prefer the term *affine geodesic* for affinely-parameterised vector fields (satisfying $\nabla_\xi \xi = 0$).

$\psi = 0$, so that ξ is hypersurface-orthogonal. Since ξ is both null and hypersurface-orthogonal, it is a wave vector field. Any gravitational wave thus admits a class of conformally related wave vector fields. Now, Lemma 2.1.3 ensures that there exists a function κ such that $\nabla_{\xi'}\xi' = \kappa\xi'$. In terms of ξ, this equality becomes $f\nabla_\xi\xi = \kappa\xi - \xi\,[f]\,\xi$. Choosing f such that $\xi\,[f] = \kappa$, one obtains $\nabla_\xi\xi = 0$ and ξ is thus affine geodesic. □

Without loss of generality, the wave vector field ξ will be taken to be affine geodesic from now on. The equation $\nabla_\xi\xi = 0$ implies that $\mathcal{L}_\xi\Omega = 0$ (as can be obtained from the local expression of the curl of the wave covector, $\partial_{[\mu}\xi_{\nu]} = \partial_{[\mu}\Omega\,\partial_{\nu]}f$, expressed in terms of Koszul connections and contracted with the vector field ξ^μ). As mentioned above, the factor Ω is thus independent of the affine parameter u along rays.

This property is important in order for Ω to acquire the interpretation of a time unit on the quotient manifold defined as follows:

Definition 2.1.5. *The Platonic screen is the orbit space of rays for a gravitational wave, i.e. the points of the Platonic screen are identified with the rays of the gravitational wave.*

There is no canonical realisation of the Platonic screen as a submanifold of the gravitational wave since various slicings are perfectly legitimate. However, any such slicing corresponds to a specific choice of representative in each orbit. These subtleties justify the rather abstract but geometric definition of the Platonic screen. A *screen worldvolume* is a submanifold of a gravitational wave providing a complete set of representatives of the Platonic screen. In other words, the points of a screen worldvolume are representatives of equivalence classes constituted by the rays (Table 2.1). In some sense, any screen worldvolume can be seen as a concrete realisation of the abstract Platonic screen (Fig.1.1).

Lemma 2.1.6. *The Platonic screen is endowed with a locally synchronisable absolute clock. The absolute time on any screen worldvolume is induced from the retarded time of the gravitational wave.*

Proof: The absolute clock locally reads $\psi = \Omega(t,x)dt$ which is well-defined on the Platonic screen, in the sense that it does not depend on the choice of screen worldvolume since the time unit Ω does not depend on the affine parameter u, as was shown previously. □

Similarly to the abstract definition of the Platonic screen, one defines a *wavefront* as the orbit space of rays of a wavefront worldvolume. Again it can also be defined more

	Spacetime	Coordinates	Structure
Manifold	Ambient spacetime	(u, t, x^i)	Lorentzian
Quotient manifold	Platonic screen	(t, x^i)	Aristotelian
Submanifold	Screen worldvolume, *e.g.* $u = 0$		

Table 2.1: Summary of the spacetimes in the ambient approach

	Leave	Coordinates	Signature
Manifold	Wavefront worlvolume t =const	(u, x^i)	Null
Quotient manifold	Wavefront	(x^i)	Riemannian
Submanifold	Screen t =const and *e.g.* $u = 0$		

Table 2.2: Summary of the leaves in the ambient approach

concretely by the intersection between a wavefront worldvolume and a screen worldvolume, intersection which will be called a screen (Fig.1.2). In other words, a *screen* is a submanifold of a wavefront worldvolume providing a complete set of representatives of the wavefront (Table 2.2). A smooth choice of representatives for the complete set of wavefront worldvolumes defines a screen worldvolume. As a side remark, let us notice that the screen worldvolumes can be of any type. When the context makes it clear, screen worldvolumes will sometimes be improperly referred to as "screen" for the sake of concision (as in Fig.2.3). For instance, the Platonic screen actually corresponds to an infinite collection of equivalent screen worldvolumes, only differing by the choice of representatives along the rays.

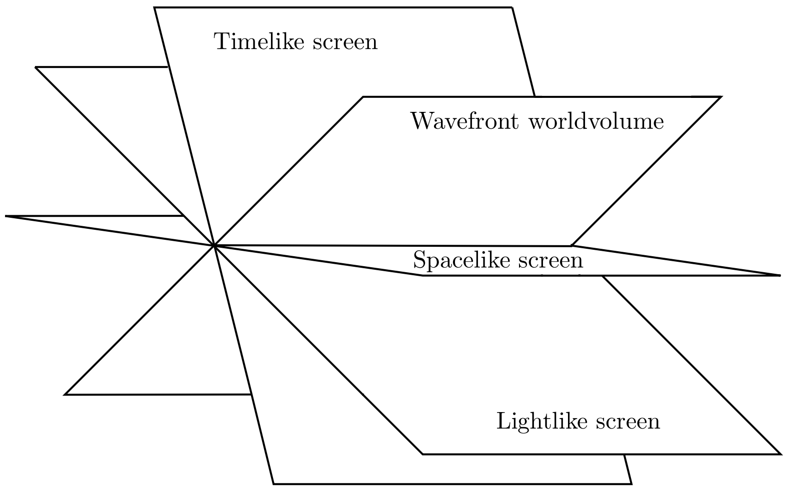


Figure 2.3: Examples of screens of different types

2.2 Definitions of Platonic gravitational waves

2.2.1 Bargmann-Eisenhart waves

Of high interest is the class of gravitational waves with parallel rays. Precisely this class of metrics was considered by Eisenhart [20] in his description of dynamical trajectories as geodesic motions, so these spacetimes are sometimes called "Eisenhart spacetimes" by mathematicians (*cf. e.g.* [22, 94]). However, the bridge between nonrelativistic physics and general relativity was rediscovered independently much later and considerably generalised in [24, 25] where such spacetimes were called "Bargmann spacetimes" in order to stress the natural appearance of the Bargmann group [26] as the structure group in this setting. Therefore, as a tribute to both prestigious men, we will refer to these spacetimes as "Bargmann-Eisenhart"[7].

Definition 2.2.1. *A Bargmann-Eisenhart wave is a Lorentzian structure with a parallel null vector field.*

In this subsection, the ambient metric will be denoted $\bar{g}$ in agreement with the line element (1.3.13). As suggested by our choice of terminology, these spacetimes are indeed gravitational waves. This can easily be seen as follows. The null vector field, being parallel, is necessarily curl-free and then the associated 1-form $\bar{g}(\xi)$ is closed; thus, ξ is (trivially) hypersurface-orthogonal. Therefore any parallel null vector field is a wave vector field and the wave covector field is closed.

7. As a side historical remark, these spacetimes were considered by [21] so they are also sometimes called "Brinkmann" spacetimes [95].

Example: It looks somehow natural to look for examples among maximally symmetric spacetimes, but this is deceptive because Minkowski spacetime is the only maximally-symmetric Bargmann-Eisenhart wave. Indeed, spacetimes with a nonvanishing constant curvature do not admit parallel vector fields.

Since Bargmann-Eisenhart waves are gravitational waves, one can use the Brinkmann coordinates in order to bring their line element in its canonical form [8]. Following the prescription sketched in Section 2.1, one identifies $\frac{\partial}{\partial u}$ with the null vector field ξ. Being parallel, ξ is also Killing and one has $\mathcal{L}_\xi \bar{g} = 0$, that is, all components of the metric $\bar{g}$ are independent of the coordinate u. Furthermore, locally $\bar{g}\,(\xi) = df$ (since the wave covector field is closed) and, identifying the phase f with the coordinate t, one obtains $\Omega = 1$. The line element of a Bargmann-Eisenhart wave then takes the canonical form:

$$\begin{aligned} d\bar{s}^2 &= \bar{g}_{tt}(t,x)\,dt^2 \,+\, 2\,dtdu + 2\bar{g}_{ti}(t,x)\,dx^i dt \,+\, \bar{g}_{ij}(t,x)\,dx^i dx^j \\ &= 2\,dt\,\big(du + \bar{A}_i(t,x)\,dx^i - \bar{U}(t,x)dt\big) \,+\, \bar{g}_{ij}(t,x)\,dx^i dx^j \end{aligned} \tag{2.2.6}$$

where in the second equation one introduced the *scalar potential* $\bar{U} = -\frac{1}{2}\bar{g}_{tt}$, the *Coriolis 1-form* $\bar{A}_i = \bar{g}_{ti}$ (also called *vector potential*) and the spatial metric $\bar{g}_{ij}$. This choice of terminology essentially follows the common usage in the Bargmann framework [32]. We will also refer to the coordinate t, that is the primitive of the parallel null vector field as the absolute time (called "Galilean" time in [24, 25]), because of its nonrelativistic interpretation in the Aristotelian structure. On flat spacetime $\big(\bar{U} = \bar{A}_i = 0,\ \bar{g}_{ij} = \delta_{ij}\big)$, the absolute time is identified with the light-cone time which is a null coordinate but one should keep in mind that, in general, the coordinate vector field $\partial/\partial t$ corresponding to the absolute time itself can be of any type. The arbitrariness of the signature of the screen worlvolume $u = 0$ befalls to the arbitrariness of the type of $\partial/\partial t$, as can be seen from the screen worldvolume line element (1.3.6). It is quite remarkable that the ambient spacetime, obtained from a nonrelativistic spacetime by adding an extra coordinate u and endowed with line element (2.2.6), has always a Lorentzian signature, despite the arbitrariness on the type of the direction t.

The canonical form of the line element is preserved by local Abelian gauge transformations along the null fiber ($u \mapsto u - \Lambda(t,x)$, $\bar{U} \mapsto \bar{U} - \partial_t\Lambda$, $\bar{A}_i \mapsto \bar{A}_i + \partial_i\Lambda$) and by coordinate transformations of the last $d = n-1$ coordinates ($x^i \mapsto {x'}^i(t,x)$, $\bar{U} \mapsto \bar{U} - \frac{1}{2}\bar{A}_i\frac{\partial x^i}{\partial t'} - \bar{g}_{ij}\frac{\partial x^i}{\partial t'}\frac{\partial x^j}{\partial t'}$, $\bar{A}_i \mapsto \bar{A}_j\frac{\partial x^j}{\partial x^{i'}} + \bar{g}_{kl}\frac{\partial x^k}{\partial t'}\frac{\partial x^l}{\partial x^{i'}}$, $\bar{g}_{ij} \mapsto \bar{g}_{kl}\frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^l}{\partial x^{j'}}$). While the second transformations correspond to coordinate transformations on the wavefronts, the first transformations correspond to the arbitrariness in the choice of the origin for the affine parameter along the rays. Phys-

8. We closely follow the discussion in the Section 2.2 from the lecture notes [93].

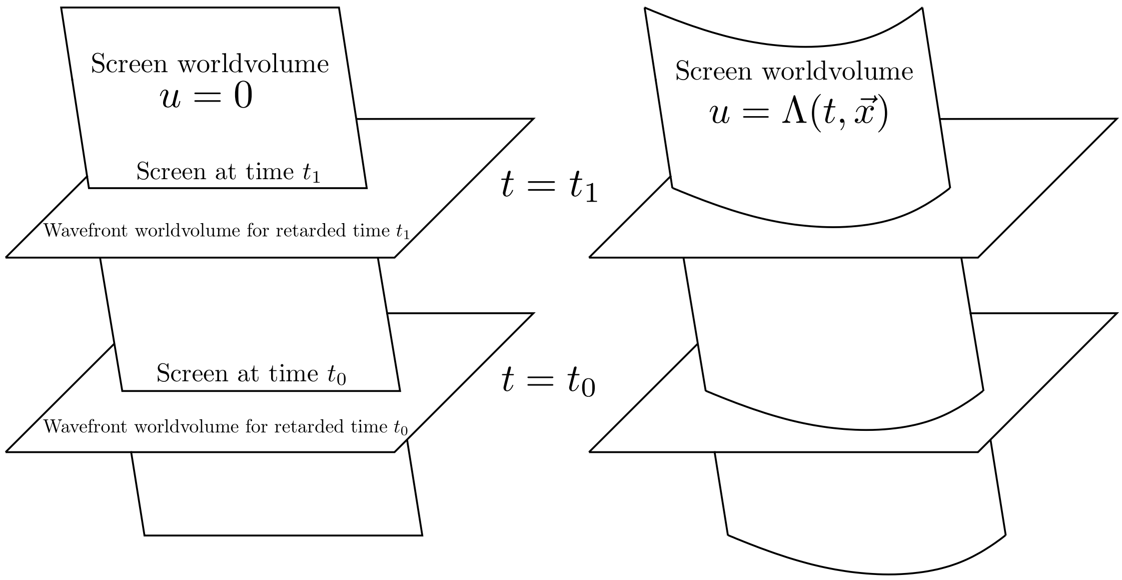


Figure 2.4: Gauge transformation of u relate different choices of screen worldvolume.

ically, these transformations correspond to different choices of the screen worldvolume, from say the hypersurface $u = 0$ to the hypersurface $u' = u - \Lambda(t, x) = 0$ (*cf.* Fig. (2.4)). Let us point out that the previous transformations also have a nonrelativistic interpretation. For instance, the Abelian gauge transformations correspond to equivalence relation between Lagrangians differing by a total derivative, as mentioned at the end of Section 1.3. Moreover, the coordinate transformations on the wavefronts correspond to the reparameterisation (1.3.2) of holonomic coordinates.

Furthermore, locally, it can be shown (*cf.* *e.g* the Section 10.1 of [10]) that one of the potentials, either the scalar or the vector one, can be put to zero by a suitable coordinate transformation:

$$
\begin{aligned}
u &= u' + f\left(t', x'\right) , \\
x^i &= x^i\left(t', x'\right) ,
\end{aligned}
$$

corresponding to the following redefinitions

$$
\bar{U}' = \bar{U} - \frac{\partial f}{\partial t'} - \frac{1}{2}\bar{A}_i \frac{\partial x^i}{\partial t'} - \bar{g}_{ij}\frac{\partial x^i}{\partial t'}\frac{\partial x^j}{\partial t'} , \tag{2.2.7}
$$

$$
\bar{A}'_i = \frac{\partial f}{\partial x^{i'}} + \bar{A}_j \frac{\partial x^j}{\partial x^{i'}} + \bar{g}_{kl}\frac{\partial x^k}{\partial t'}\frac{\partial x^l}{\partial x^{i'}} , \tag{2.2.8}
$$

$$
\bar{g}'_{ij} = \bar{g}_{kl}\frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^l}{\partial x^{j'}} . \tag{2.2.9}
$$

It seems plausible that in fact both potentials can be set to zero, $\bar{U}' = \bar{A}'_i = 0$, as is natural since we have as many arbitrary functions (f and x^i) as potentials ($\bar{U}$ and $\bar{A}_i$) at our disposal; however, we are not aware of any rigorous proof of this expectation.

The curvature two-form $\bar{F}_{ij} = \partial_{[i}\bar{A}_{j]}$ of the Coriolis 1-form is called the *Coriolis two-form*. A Bargmann-Eisenhart wave whose Coriolis 1-form vanishes will be called *Coriolis-free*.

Let us turn back now to nonrelativistic structures and see in which sense the Platonic screen of a Bargmann-Eisenhart wave is a nonrelativistic spacetime with an Aristotelian structure. As mentioned earlier, the wavefront worldvolumes of a gravitational wave are not absolute spaces since they are null hypersurfaces (the induced metric ${}^{\perp}\gamma$ is degenerate on the wavefront worldvolumes) and so although gravitational waves may induce a (locally synchronisable) absolute clock on the wavefront worldvolumes, they lack the necessary structure to define absolute spaces. However, the wavefronts of Bargmann-Eisenhart waves are Riemannian manifolds (so the Platonic screen possesses an absolute space). In order to see why, notice that since the coordinate vector fields $\frac{\partial}{\partial x^i}$ are orthogonal to $\frac{\partial}{\partial u}$, the (induced) metric on a wavefront is well-defined on the orbits. The tangent vectors to the wavefront are equivalence classes $[v]$ of vectors $v \sim v + \alpha\,\xi$ ($\alpha \in \mathbb{R}$) and the Killing property of ξ ensures that the induced metric is constant along rays. Very concretely, the components of the positive-definite metric on the wavefronts read $\bar{g}_{ij}(t,x)$ in the Brinkmann coordinates.

We now reformulate the very beginning of the Section 2 in [24] with our own terminology:

Lemma 2.2.2. *The Platonic screen of a Bargmann-Eisenhart wave is a nonrelativistic spacetime, where the Aristotelian*[9] *structure is induced from the ambient metric.*

Let us now focus on a subclass of Bargmann-Eisenhart spacetimes introduced by Brinkmann [21] and vividly studied since: the so-called pp-waves. A gravitational wave is *plane-fronted* if the wavefronts define an absolute space which is flat. Similarly, a *Lobachevsky-plane-fronted wave* is a gravitational wave where wavefronts are Lobachevsky planes [96] (or hyperbolic spaces in higher dimensions).

Definition 2.2.3. *The term pp-wave stands for plane-fronted wave with parallel rays (or propagation) and designates a spacetime admitting a parallel null vector field such that the wavefronts are flat.*

A widespread – though slightly misleading – terminology defines pp-waves as what we called Bargmann-Eisenhart waves (*cf.* *e.g.* Section 10.1 of [10]). The reason behind

9. More precisely, the induced structure is Augustinian (*cf.* Definition 3.2.16) since the absolute clock can be shown to be closed.

this choice of terminology is the fact it implicitly assumes that only solutions of vacuum Einstein equations are considered. Indeed, Bargmann-Eisenhart waves which are Ricci-flat are plane-fronted in "low" dimensions $D \leqslant 5$, since they have a Ricci-flat spatial metric which, for $d \leqslant 3$, is consequently flat. Moreover, Ricci-flat pp-waves are (essentially, *cf.* discussion below) Coriolis-free. Presumably for this reason, pp-waves in the sense of the literal Definition 2.2.3 were called "gyratons" in [97]. As suggested by this terminology, (nonvanishing) Coriolis covector field somewhat encodes gyroscopic effects.

In a Brinkmann coordinate system with Cartesian coordinates on the wavefront, the line element of a pp-wave takes the canonical form:

$$ds^2 \;=\; 2\,dt\left(du + \bar{A}_i(t,x)\,dx^i - \bar{U}(t,x)dt\right) \,+\, a^{-2}(t)\delta_{ij}\,dx^i dx^j \tag{2.2.10}$$

since here each wavefront is a flat Riemannian manifold by assumption, *i.e.* the metric $\bar{g}_{ij}(t,\vec{x})$ is flat for fixed absolute time t. However, the coordinate transformation $\vec{x}' = a^{-1}\vec{x}$ preserving the canonical form of the metric allows us to assume without loss of generality that the canonical form of the pp-wave metric is

$$ds^2 \;=\; 2\,dt\left(du' + \bar{A}'_i(t,\vec{x}')\,dx'^i - \bar{U}'(t,\vec{x}')dt\right) \,+\, \delta_{ij}\,dx'^i dx'^j. \tag{2.2.11}$$

We now establish that Einstein pp-waves are (under topological assumptions on the wavefront) Coriolis-free. We start by noting that the Ricci scalar of a Bargmann-Eisenhart wave is equal to the one of the wavefront, and therefore vanishes for pp-waves. Einstein pp-waves are then necessarily Ricci-flat. We now establish the following Lemma:

Lemma 2.2.4. *When the first Betti number of its wavefront is zero, a gravitational wave with zero Coriolis force ($\bar{F} = 0$) is Coriolis-free.*

Proof: If the Coriolis force vanishes, then the Coriolis 1-form is closed ($\bar{F} = \bar{d}\bar{A} = 0$) with respect to the spatial de Rham differential $\bar{d} := dx^i\partial_i$. Furthermore, if the first Betti number of the wavefront is zero, then the Coriolis 1-form is exact ($\bar{A} = \bar{d}f$) and then can be gauged away via a local abelian transformation along the fiber. □

Making use of this Lemma, we establish the following Proposition:

Proposition 2.2.5. *When the first and second Betti numbers of its wavefront are zero, an Einstein pp-wave is Coriolis-free.*

Proof: The spatial 2-form $\bar{F}$ on the wavefront is exact by definition ($\bar{F} = \bar{d}\bar{A}$), thus it is closed ($\bar{d}\bar{F} = 0$). The Ricci equation $R_{-i} = 0$ implies that $\bar{F}$ is also coclosed

($\star\,\bar{d}\star\bar{F}=0$). When the second Betti number of the wavefront is zero, there are no harmonic 2-forms on it. Therefore, the Coriolis curvature is vanishing. When the first Betti number of the wavefront is zero, this implies the Coriolis-freeness. □

Coriolis-free pp-waves then occupy a distinguished place among Bargmann-Eisenhart spacetimes. In fact, we can show that Coriolis-free pp-waves are Kerr-Schild spacetimes, a class of metrics we now briefly review. We will refer to a (generalised) Kerr-Schild spacetime as a manifold endowed with a metric of the following form: $g_{\mu\nu}=\mathfrak{g}_{\mu\nu}-2\,\mathfrak{U}\,\xi_\mu\xi_\nu$, with ξ a null vector field and $\mathfrak{g}_{\mu\nu}$ a constant curvature background. In flat four dimensional spacetime, this class was studied in [98] by Kerr and Schild, and was generalised to higher dimensions in [99] and to (A)dS backgrounds in [100] where the following properties have been shown in full generality:

• The inverse metric takes the (exact) form: $g^{\mu\nu}=\mathfrak{g}^{\mu\nu}+2\,\mathfrak{U}\,\xi^\mu\xi^\nu$ (and $|g|=1$ for flat background).

• The vector field ξ is null or geodesic (or even affine geodesic) equivalenty with respect to g or $\mathfrak{g}$.

• The expansion, shear and twist are the same with respect to g or $\mathfrak{g}$.

• If the potential $\mathfrak{U}$ of a Kerr-Schild spacetime is constant along the affine geodesic null vector field, then the latter is Killing (or even parallel) equivalently with respect to g or $\mathfrak{g}$.

From the above canonical form, we see that Coriolis-free pp-waves $\big(\bar{A}_i=0\big)$ are Kerr-Schild spacetimes with Minkowski background metric: $g_{\mu\nu}=\eta_{\mu\nu}-2\mathfrak{U}\xi_\mu\xi_\nu$. In Brinkmann coordinates the Minkowski metric reads $ds^2=2\,dtdu+d\vec{x}^2$, while the Kerr-Schild potential is identified with the pp-wave potential $\mathfrak{U}\equiv\bar{U}$ and $\xi=\frac{\partial}{\partial u}$ is the null parallel vector field.

A well-known property of the Kerr-Schild spacetimes is the fact that their fully nonlinear Einstein equations reduce to their linearisation around the background metric $\mathfrak{g}$ *i.e.* Kerr-Schild spacetimes linearise the Einstein tensor. This feature greatly simplifies the equations of motion. Accordingly, for Coriolis-free pp-waves the vacuum Einstein equations reduce to the linear Laplace equation for the potential $\bar{U}$ and, as such, Coriolis-free pp-waves traveling along the same direction are seen to obey to a superposition principle.

Examples of Coriolis-free pp-waves:

• An *exact plane wave* is a Coriolis-free pp-wave whose scalar potential is a quadratic form in the Cartesian coordinates x^i. The line element of an exact plane wave then takes the form:

$$ds^2 \quad = \quad 2\,dt\left(du - M_{ij}\,(t)\,x^i x^j dt\right) \,+\, \delta_{ij}\,dx^i dx^j \tag{2.2.12}$$

with $M_{ij}(t)$ an arbitrary symmetric $d \times d$ matrix.

A *homogeneous plane wave* is an exact plane wave whose quadratic form is independent of the absolute time. A homogeneous plane wave whose matrix M is proportional to the identity is a *homogeneous pp-wave (Hpp-wave).* Hpp-waves have been studied in the null dimensional reduction framework in [46] where they were shown to induce nonrelativistic spacetimes with cosmological constant (Newton-Hooke spacetimes), whose symmetry group is that of the harmonic oscillator.

Exact plane waves are well known to enjoy the following two properties:

- An exact plane wave is conformally flat if and only if it is a Hpp-wave. Indeed, the only nonvanishing component of the Weyl tensor of a Coriolis-free pp-wave reads, in Brinkmann coordinates $C_{-i-j} = \partial_i\partial_j\bar{U} - \frac{1}{d}\delta_{ij}\partial_k\partial^k\bar{U}$. Substituting $\bar{U} = M_{ij}x^i x^j$, one obtains the following condition for the matrix M in order for the exact plane wave to be conformally-flat: $M_{ij} = \frac{1}{d}\delta_{ij}M^k_k$ and M is therefore proportional to the identity: $M_{ij} = \alpha\,(t)\,\delta_{ij}$ with α an arbitrary function of t. The graph of the potential of a conformally-flat plane gravitational wave is therefore a paraboloid of revolution.
- The most important property of exact plane waves, that gave their name, is that they are Einstein manifolds if and only if their quadratic form is traceless.
  As we noted, demanding that a pp-wave is an Einstein manifold is then equivalent for it to be Ricci-flat. The only nonvanishing component of the Ricci tensor of a Coriolis-free pp-wave in Brinkmann coordinates reads: $R_{--} = \partial_k\partial^k\bar{U}$. Substituting $\bar{U} = M_{ij}x^i x^j$, we see that the Ricci-flat condition is satisfied if and only if M is traceless. A traceless symmetric $d \times d$ matrix indeed parameterises the (transverse) polarisation states of an on-shell *linearised* gravitational wave. We saw that this property remains manifest at nonlinear level for the Ricci-flat plane gravitational wave.

2.2.2 Platonic waves as conformal Bargmann-Eisenhart waves with preserved null Killing vector

The following definition of a Platonic wave is motivated by the most general form (1.3.14) of the line element for which the null dimensional reduction works. Its goal is to explain the geometric origin of the line element considered by Lichnerowitz [22] and their relation with Bargmann-Eisenhart waves. Later on, an equivalent definition will be provided that displays an explanation for the fact that their Platonic screen carries a structure of nonrelativistic spacetime.

Definition 2.2.6. *Platonic waves are Lorentzian structures with a null Killing vector field such that the latter becomes parallel with respect to a conformally equivalent metric.*

As suggested by our choice of terminology, they are indeed gravitational waves: their null Killing vector field is a wave vector field, as explained below. The definition should be understood in more concrete terms as follows: let ξ denote the null Killing vector field with respect to the metric g, *i.e.* $\mathcal{L}_\xi g = 0$. The further hypothesis is that there exists a conformally related metric $\bar{g}$, that is to say $g = \Omega\bar{g}$, such that $\bar{\nabla}\xi = 0$, where $\bar{\nabla}$ is the Koszul connection with respect to $\bar{g}$.

As is clear from the previous definition, a Platonic wave is conformally related to a Bargmann-Eisenhart wave, both sharing the same null Killing vector field ($\mathcal{L}_\xi g = 0 = \mathcal{L}_\xi \bar{g}$) since a parallel vector field is automatically Killing. Hence a number of properties of Platonic waves will be easily derived from those of Bargmann-Eisenhart manifolds. Obviously, any Bargmann-Eisenhart wave is trivially a Platonic wave.

Examples: It is natural to look again for examples among maximally symmetric spacetimes. Minkowski spacetime is of course a Platonic wave since it is even a Bargmann-Eisenhart wave. Surprisingly enough, de Sitter spacetime is *not* a Platonic wave since it does not admit a Killing vector field which is globally null (not only at the Killing horizon). So the simplest example of a proper Platonic wave ("proper" in the sense that it is *not* a Bargmann-Eisenhart wave) is anti de Sitter spacetime.

Before writing the canonical form of the Platonic metric in Brinkmann coordinates, we first check that Platonic waves are gravitational waves. The proof rests on the one for Bargmann-Eisenhart waves, where we established that the 1-form dual to the null vector field ξ by the Bargmann-Eisenhart metric $\bar{g}$ is locally exact: $\bar{g}\,(\xi) = df$. Therefore, the 1-form obtained via the conformally related metric $g = \Omega\,\bar{g}$ writes locally $g\,(\xi) = \Omega\,df$ and ξ indeed is hypersurface-orthogonal.

For later purposes, let us establish the following facts:

Lemma 2.2.7. *Two conformally equivalent spacetimes possess the same Killing vector field if and only if the conformal factor is constant along this vector field.*

Proof: The proof is quite straightforward: one makes use of the vanishing of the Lie derivative of the metric along a Killing vector field and of the Leibniz rule. This implies that the conformal factor Ω satisfies $\mathcal{L}_\xi \Omega = 0$ (similarly to $\mathcal{L}_\xi f = 0$ for any gravitational wave). □

Proposition 2.2.8. *For any Platonic wave:*

- *The conformal factor that relates it to a Bargmann-Eisenhart spacetime is constant along the null Killing vector field.*
- *The null Killing vector field is hypersurface-orthogonal and its integrating factor is equal to the conformal factor. So both the primitive and the integrating factor are constant along the null Killing vector field.*

Proof: A vector field is parallel if and only if it is Killing (so the Lemma implies the first point) and curl-free (which shows the second point, since hypersurface-orthogonal is equivalent to conformally-curl-free). □

This justifies the use of Brinkmann coordinates and explains the form of canonical line element of Platonic waves:

$$\begin{aligned} ds^2 &= g_{tt}(t,x)\,dt^2 + 2\,\Omega(t,x)\,dtdu + 2g_{ti}(t,x)\,dx^i dt + g_{ij}(t,x)\,dx^i dx^j \\ &= \Omega\,(t,x)\left[2\,dt\left(du + \bar{A}_i(t,x)\,dx^i - \bar{U}(t,x)dt\right) + \bar{g}_{ij}(t,x)\,dx^i dx^j\right]. \end{aligned} \quad (2.2.13)$$

The second equation emphasises the interpretation of Platonic waves as conformal Bargmann-Eisenhart waves. In order to obtain this canonical form, one can also repeat the argument used in Section 2.1.3 and use the independence of the Platonic metric from the coordinate u since it corresponds to a Killing direction.

Remark: A spacetime conformally equivalent to a Bargmann-Eisenhart wave via a conformal factor that only depends on the absolute time is itself a Bargmann-Eisenhart wave admitting the same null parallel vector. As shown in [25], the converse is also true: two Bargmann-Eisenhart waves are conformally equivalent if and only if the conformal factor that relates them only depends on the absolute time. The metric of such a spacetime (with conformal factor $\Omega(t)$) can always be put in the canonical form (2.2.6) via a redefinition of t of the form $t \mapsto t' = \int^t \Omega(\tau)d\tau$, $dt' = \Omega(t)dt$:

$$\begin{aligned} ds^2 &= \Omega(t)\,\left[\bar{U}(t,x)\,dt^2 + 2\,dtdu + 2\bar{A}_i(t,x)\,dx^i dt + \bar{g}_{ij}(t,x)\,dx^i dx^j\right] \\ &= 2\,dt'\left(du + \bar{A}'_i(t',x)\,dx^i - \bar{U}'(t',x)dt'\right) + \bar{g}'_{ij}(t',x)\,dx^i dx^j \end{aligned}$$

with $\bar{U}'(t',x) = \Omega^{-1}(t)\bar{U}(t,x)$, $\bar{A}'_i(t',x) = \bar{A}_i(t,x)$ and $\bar{g}'_{ij}(t',x) = \Omega(t)\bar{g}_{ij}(t,x)$.

2.2.3 Platonic gravitational waves as Julia-Nicolai spacetimes

We now show the equivalence between the Platonic waves introduced in the previous subsection and the class of spacetimes studied by Julia and Nicolai in [72].

To do so, we proceed in two steps: firstly, by reviewing the equivalence between spacetimes satisfying the Julia-Nicolai condition and gravitational waves with a Killing wave vector field and, secondly, by showing the equivalence between the latter class and the one of Platonic waves.

In [72], the authors focused on a class of Lorentzian structures which admit a null Killing vector field and which are solutions of the vacuum Einstein equations. In the following, we will consider spacetimes satisfying the *Julia-Nicolai condition*: $R(\xi, \xi) := R_{\mu\nu}\xi^\mu\xi^\nu = 0$, with R the Ricci tensor, without the further assumption that the spacetimes considered are Einstein, as the other components of the vacuum Einstein equations play no role in the argument.

Lemma 2.2.9 (Julia-Nicolai [72])**.** *A Lorentzian structure admitting a null Killing vector field satisfies the Julia-Nicolai condition if and only if the null Killing vector field is hypersurface-orthogonal.*

In order to be self-contained, we review the proof presented in the Section 2 of [72] (here in arbitrary [10] dimension) and complete some steps that were left to the reader.

Proof: By contracting the commutator of two Koszul connections of the 1-form $\psi \equiv g(\xi)$, by ξ and then contracting the indices, we easily see that the Julia-Nicolai condition is equivalent to $\xi\left(\nabla^2\psi\right) := \xi^\mu\left(g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi_\mu\right) = 0$ if ξ is Killing. Furthermore we have, for any Killing vector field ξ with constant norm, the equivalence:

$$\xi\left(\nabla^2\psi\right) = 0 \Leftrightarrow (d\psi)^2 = 0$$

with $(d\psi)^2 := (d\psi)_{\mu\nu}(d\psi)^{\mu\nu}$. We now prove the following Lemma:

Lemma 2.2.10. *For ξ a null affine geodesic vector field with dual 1-form $\psi \equiv g(\xi)$ the following equivalence holds:*

$(d\psi)_{\mu\nu}(d\psi)^{\mu\nu} = 0 \Leftrightarrow \psi \wedge d\psi = 0.$

In order to establish this Lemma, we place ourselves in an adapted frame, such that the only nonvanishing component of the 1-form is $\psi_+ \neq 0$.

The vector ξ, being affine geodesic and null, satisfies $\xi(d\psi) = 0$ which reduces in an adapted frame to $(d\psi)_{a-} = 0$ and we then have $(d\psi)^2 = (d\psi)^{ij}(d\psi)_{ij}$. The condition $(d\psi)^2 = 0$ is then equivalent to $(d\psi)_{ij} = 0$. On the other hand, the only nontrivial component of $\psi \wedge d\psi$ in this frame is $(\psi \wedge d\psi)_{+ij} = \psi_+(d\psi)_{ij}$ which also vanishes if and only if $(d\psi)_{ij} = 0$, concluding the proof.

We therefore established the following string of equivalences:

10. An alternative proof that a hypersurface-orthogonal vector field satisfies the Julia-Nicolai condition via the four-dimensional Raychaudhuri's equation can be found in [101].

$R(\xi,\xi) = 0 \Leftrightarrow \xi\left(\nabla^2\psi\right) = 0 \Leftrightarrow (d\psi)^2 = 0$ for a Killing vector with constant norm and $(d\psi)^2 = 0 \Leftrightarrow \psi \wedge d\psi = 0$ which stands for a affine geodesic null vector field.

Remembering that the constant norm and affine geodesic conditions are satisfied by a null Killing vector field allows to write $R(\xi,\xi) = 0 \Leftrightarrow \psi \wedge d\psi = 0$ for a null Killing vector field. Using Frobenius Theorem concludes the proof. □

We already showed in Section 2.2.2 that Platonic waves are gravitational waves. By definition, they possess a wave Killing vector field. Our next task concerns the equivalence of the class of Platonic waves with the class of gravitational waves with a Killing wave vector field.

Proposition 2.2.11. *A gravitational wave possesses a wave vector field that is Killing if and only if it is a Platonic wave.*

Proof: Starting from a spacetime characterised by the metric g and admitting a Killing wave vector field ξ (*i.e.* $\mathcal{L}_\xi g = 0$) whose dual 1-form locally reads $g(\xi) = \Omega\, df$, we consider a conformally related metric $\bar{g}$ via the integrating factor Ω, that is $g = \Omega\bar{g}$. Computing the Lie derivative $\mathcal{L}_\xi g = \mathcal{L}_\xi\Omega\,\bar{g} + \Omega\,\mathcal{L}_\xi\bar{g}$ and recalling from Section 2.1.3 that the integrating factor Ω of an affine geodesic wave vector field is constant along this vector field ($\mathcal{L}_\xi\Omega = 0$), we conclude that ξ is Killing for both metrics. Furthermore, the dual 1-form associated to ξ via $\bar{g}$ reads $\bar{g}(\xi) = df$, so the vector field ξ is curl-free with respect to the metric $\bar{g}$. Being both Killing and curl-free, ξ is parallel with respect to $\bar{\nabla}$ and thus, we have shown that the initial spacetime admitting a null Killing vector field is conformally related to a spacetime with respect of which this same vector becomes parallel. In other words, it is a Platonic wave. □

From the point of view of the ambient approach, the definition of Platonic waves as gravitational waves with a Killing wave vector field is somewhat the most natural requirement for the wavefronts to define an absolute space. Indeed, the wavefront worldvolumes are null hypersurfaces but the corresponding wavefronts or, equivalently screens, are Riemannian manifolds. The proof of this fact follows exactly the same steps as for the case of Bargmann-Eisenhart waves whose crucial ingredient was the Killing property which ensures that the metric does not depend on the choice of screen worldvolume. In other words, the Platonic waves are the most general class of gravitational waves such that their Platonic screen is canonically endowed with an Aristotelian structure [11].

Proposition 2.2.12. *The Platonic screen of a Platonic wave is a nonrelativistic spacetime, where the Aristotelian structure is induced from the ambient metric.*

11. Strictly speaking, the most general class is the class of Kundt waves (*cf.* Section 4.2). More accurately, the Platonic waves are the most general waves inducing an Aristotelian structure on their space of Killing orbits.

In other words, the nonrelativistic structure of the Platonic screen is the shadow of the relativistic structure of the Platonic wave. In a Brinkmann chart, the validity of the Proposition is manifest since the absolute clock and space are respectively defined by:

$$\psi = \Omega(t,x)dt\,, \qquad d\ell^2 = g_{ij}(t,x)dx^i dx^j\,.$$

2.2.4 Platonic gravitational waves as Kundt spacetimes

We conclude this Section by showing that Platonic waves belong to the Kundt class (introduced in [102], see [103] for a detailed account), in the following sense [12]:

Definition 2.2.13. *A Kundt wave is a Lorentzian structure possessing a null geodesic, expansionless, shearless and twistless vector field.*

In other words, the three optical scalars of the gravitational wave must vanish.

Lemma 2.2.14. *Platonic waves are Kundt waves.*

This property will play an important role in the characterisation of Platonic waves (*cf.* Section 2.4.2) since the classification of Kundt waves in any dimension has recently been developed extensively [103].

Proof: We already know that the null Killing vector field ξ characterising a Platonic wave is hypersurface-orthogonal and geodesic. Besides, being null and Killing, the vector field ξ is necessarily affine geodesic, allowing the use of the following Lemma (for a proof, see [104] Section 2.4.3):

Lemma 2.2.15. *Consider an affine geodesic vector field ξ, then ξ is hypersurface-orthogonal if and only if its twist vanishes.*

Therefore the vector field ξ is twistless. Furthermore, being Killing, it is also expansionless and shear-free. □

Remark: The Kundt property implies that the second fundamental form (also called extrinsic curvature) on the wavefront worldvolumes vanishes; thus, the latter are totally geodesic.

The general form of Kundt metrics reads [105]:

$$d\tilde{s}^2 = 2dt\left(du - \tilde{U}\left(u,t,x\right)dt + \tilde{A}_i\left(u,t,\vec{x}\right)dx^i\right) + \tilde{g}_{ij}\left(t,x\right)dx^i dx^j. \tag{2.2.14}$$

12. As for other classes of spacetimes, the terminology is a bit fuzzy in the literature because of the fact that often they are implicitly assumed to be solutions of Einstein equations (*e.g.* Section 27.1 of [101]). We adopt a geometric definition which is used for instance in [103].

From this canonical form of the line element, it is manifest that (i) Kundt waves are gravitational waves and (ii) Bargmann-Eisenhart waves belong to the Kundt class. The first assertion can morevover be refined as:

Proposition 2.2.16. *A gravitational wave is a Kundt wave if and only if, in Brinkmann coordinates, the wavefront metric $\bar{g}_{ij}$ is independent of the coordinate u.*

Proof: The proof is straightforward by performing the redefinition $u \mapsto \Omega(t,x)\, u$ in 2.2.14 and comparing with the line element 2.1.1. □

However, the previously shown fact that Platonic waves belong to the Kundt class is less transparent from this point of view and requires additional work to make link between the canonical form of the line element for a Platonic wave (2.2.13) and the one for a Kundt wave (2.2.14). Starting from the Platonic line element (2.2.13) and performing the redefinition $u' = \Omega\, u$ puts the Platonic metric in the Kundt form (2.2.14) with $\tilde{U}(u', t, x) = \Omega(t,x)\,\bar{U}(t,x) + u'\partial_t(\ln\Omega)$ and $\tilde{A}_i(u', t, x) = \Omega\bar{A}_i(t,x) - u'\partial_i(\ln\Omega)$. The potential and Coriolis form acquire a linear dependence in u' and then Platonic waves are seen to belong to the more restrictive class of *degenerate* Kundt spacetimes [103] for which the potential and Coriolis form of (2.2.14) take the specific form [13]:

$$\begin{aligned} \tilde{U}(u,t,x) &= u^2\tilde{U}^{(2)}(t,x) + u\,\tilde{U}^{(1)}(t,x) + \tilde{U}^{(0)}(t,x) \\ \tilde{A}_i(u,t,x) &= u\,\tilde{A}_i^{(1)}(t,x) + \tilde{A}_i^{(0)}(t,x)\,. \end{aligned} \tag{2.2.15}$$

By comparison with the transformed $\tilde{U}$ and $\tilde{A}_i$, we see that for a Platonic wave brought in the canonical degenerate Kundt form (2.2.14)-(2.2.15), we have $\tilde{U}^{(2)} = 0$, $\tilde{U}^{(1)} = \partial_t(\ln\Omega)$ and $\tilde{U}^{(0)} = \Omega\bar{U}$ as well as $\tilde{A}_i^{(1)} = -\partial_i(\ln\Omega)$ and $\tilde{A}_i^{(0)} = \Omega\bar{A}_i$.

Proposition 2.2.17. *Platonic waves are degenerate Kundt waves.*

The coordinate transformations

$$\begin{aligned} u &= u'\left(\frac{\partial t}{\partial t'}\right)^{-1} + f\left(t', \vec{x}'\right) \\ t &= t\left(t'\right) \\ x^i &= x^i\left(t', \vec{x}'\right) \end{aligned}$$

13. A more geometric definition of degenerate Kundt spacetimes states that a degenerate Kundt wave has to satisfy the following two conditions: i) it must be a Kundt wave with respect to a null vector ℓ and ii) the Riemann tensor and all its Koszul connections must be of type II (or more special) in the kinematic (*i.e.* aligned with ℓ) frame, see Section 2.4.2 for terminology.

together with the redefinitions

$$\begin{aligned}
U'^{(2)}\left(t',\vec{x}'\right) &= U^{(2)}\\
U'^{(1)}\left(t',\vec{x}'\right) &= U^{(1)}\frac{\partial t}{\partial t'} - A_i^{(1)}\frac{\partial x^i}{\partial t'} + 2fU^{(2)}\frac{\partial t}{\partial t'}\\
U'^{(0)} &= \frac{\partial t}{\partial t'}\left[\left[U^{(0)} + fU^{(1)} + f^2U^{(2)}\right]\frac{\partial t}{\partial t'} + \left(\frac{\partial t}{\partial t'}\right)^{-2}\frac{\partial^2 t}{\partial t'^2} - \frac{\partial f}{\partial t'}\right.\\
&\quad \left. - \left[A_i^{(0)} + fA_i^{(1)}\right]\frac{\partial x^i}{\partial t'}\right] - \frac{1}{2}g_{ij}\frac{\partial x^i}{\partial t'}\frac{\partial x^j}{\partial t'}\\
A_i'^{(1)} &= A_j^{(1)}\frac{\partial x^j}{\partial x^{i'}}\\
A_i'^{(0)} &= \frac{\partial t}{\partial t'}\left[\frac{\partial f}{\partial x^{i'}} + \left(A_j^{(0)} + fA_j^{(1)}\right)\frac{\partial x^j}{\partial x^{i'}}\right] + g_{kl}\frac{\partial x^k}{\partial t'}\frac{\partial x^l}{\partial x^{i'}}\\
g'_{ij} &= g_{kl}\frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^l}{\partial x^{j'}}.
\end{aligned}$$

preserve the canonical form of the line element (2.2.14)-(2.2.15) for a degenerate Kundt wave. Remarkably, these transformations also preserve the subclass of Platonic waves written in the canonical form of degenerate Kundt waves in the sense that $\tilde{U}'^{(2)} = 0$, $\tilde{U}'^{(1)} = \partial'_t(\ln\Omega)$ as well as $\tilde{A}_i'^{(1)} = -\partial'_i(\ln\Omega)$. This fact will be useful in the future proof of Proposition 2.4.8.

Finally, we summarise the hierarchy of properties that have been discussed in the following chain of inclusion:

$$\begin{gathered}
\text{Gravitational waves}\\
\cup\\
\text{Kundt waves}\\
\cup\\
\text{Degenerate Kundt waves}\\
\cup\\
\text{Platonic waves}\\
\cup\\
\text{Bargmann-Eisenhart waves}\\
\cup\\
\text{pp-waves}
\end{gathered}$$

2.3 Miscellaneous Platonic waves

As an illustration, we now briefly review various types of proper Platonic waves (*i.e.* which do not belong to the Bargmann-Eisenhart class).

Anti de Sitter spacetime: the most symmetric example of a proper Platonic wave. The existence of a null Killing vector field is manifest in the Poincaré coordinates

$$ds^2 = \frac{1}{z^2}[2\,dudt + dz^2 + d\vec{y}^2]. \tag{2.3.16}$$

As one can see, the wavefronts are hyperbolic spaces of dimension d as is manifest from their line element: $d\ell^2 = \frac{1}{z^2}[dz^2 + d\vec{y}^2]$. In other words, anti de Sitter (AdS) spacetime is an example of a Lobachevsky-plane-fronted wave.

AdS-gyraton [106]: Lobachevsky-plane-fronted wave conformally equivalent to a pp-wave whose line element writes

$$ds^2 \;=\; \frac{1}{z^2}\left[2\,dt\left(du - \bar{U}(t,z,\vec{y})dt + \bar{A}_i\,(t,z,\vec{y})\right) + dz^2 + d\vec{y}^2\right]. \tag{2.3.17}$$

All curvature scalar invariants of AdS-gyratons are constant and identical to the ones of AdS.

Siklos spacetime [96]: Coriolis-free AdS-gyratons of line element

$$ds^2 = \frac{1}{z^2}\left[2\,dt\left(du - \bar{U}(t,z,\vec{y})dt\right) + dz^2 + d\vec{y}^2\right]. \tag{2.3.18}$$

This definition is related to one of the equivalent characterisation of the class of "Lobachevsky-plane gravitational wave" by Siklos himself in $D = 4$ dimensions [96]. They were later reinterpreted as "AdS pp-waves" in [107]. Siklos waves are Kerr-Schild spacetimes *i.e.* can be written as $g_{\mu\nu} = \mathfrak{g}_{\mu\nu} - 2\mathfrak{U}\xi_\mu\xi_\nu$ with $\mathfrak{g}$ the AdS metric. In Brinkmann coordinates the background metric reads (2.3.16) while the Kerr-Schild potential writes $\mathfrak{U} = z^2\,\bar{U}$ and $\xi = \frac{\partial}{\partial u}$ is the null Killing vector field. Siklos spacetimes are Einstein if and only if the scalar potential $\bar{U}$ has vanishing Laplace-Beltrami operator on AdS space, *i.e.* $\frac{1}{\sqrt{-\mathfrak{g}}}\partial_\mu\left(\sqrt{-\mathfrak{g}}\mathfrak{g}^{\mu\nu}\partial_\nu\bar{U}\right) = z^2\left(\partial_z^2\bar{U} + \partial_i\partial^i\bar{U}\right) + (2-D)\,z\partial_z\bar{U} = 0$. Einstein Siklos waves are furthermore weakly universal [108], as will be discussed in Section 2.4.2.

Kaigorodov solution [109]: Siklos spacetime with potential that only depends on the coordinate z (in the Brinkmann-Poincaré coordinates) in the following way: $\bar{U}(z) \propto z^n$ (with $D = n+1$ the dimension of spacetime). Without loss of generality, its line element is thus

$$ds^2 = \frac{1}{z^2}\left[2\,dt\,(du \pm z^n\,dt) \,+\, dz^2 \,+\, d\vec{y}^2\right]. \tag{2.3.19}$$

Kaigodorov solutions belong to the class of Einstein Siklos spacetimes. In other words, they are vacuum solutions in the presence of a negative cosmological constant.

Schrödinger spacetime ($\mathbf{Sch}_Z$): Siklos spacetime where, in the Brinkmann-Poincaré coordinates, $\bar{U}(z) \propto z^{2(1-Z)}$ where $Z \geqslant 1$ is called the *dynamical exponent* because of the nonrelativistic scale transformation $t \mapsto \lambda^Z t$, $\vec{x} \mapsto \lambda\,\vec{x}$, with $\vec{x} := (z, \vec{y})$ and $u \mapsto \lambda^{2-Z}u$, which preserves the line element

$$ds^2 = \frac{1}{z^2}\left[2\,dt\left(du + z^{2(1-Z)}dt\right) \,+\, dz^2 \,+\, d\vec{y}^2\right]. \tag{2.3.20}$$

Anti de Sitter spacetime corresponds to $Z = 1$: $\mathrm{Sch}_1 = \mathrm{AdS}$ which is the homogeneous manifold for the isometry group $O(n, 2)$ acting on its conformal boundary as conformal transformations. From the point of view of symmetries, the dynamical exponent $Z = 2$ is also of high interest: Sch_2 is a homogeneous manifold (*cf.* [33, 110] for detailed global and coordinate-independent descriptions) with the Schrödinger group $Sch(d)$ as the isometry group that acts on the conformal boundary as Schrödinger transformations (this was the property that motivated their introduction in [30, 31]). Contrary to Kaigorodov solutions, the Schrödinger spacetimes Sch_Z for $Z \neq 1$ are not solutions of Einstein equations, even in the presence of a cosmological constant. However, they are solutions of richer theories with exotic matter (such as Proca fields [30, 31]) or some supergravity theories (*cf.* *e.g.* references in [110]).

We summarise in figure 2.5 the main class of Platonic examples whose physical interest is well established by the considerable literature dwelled upon.

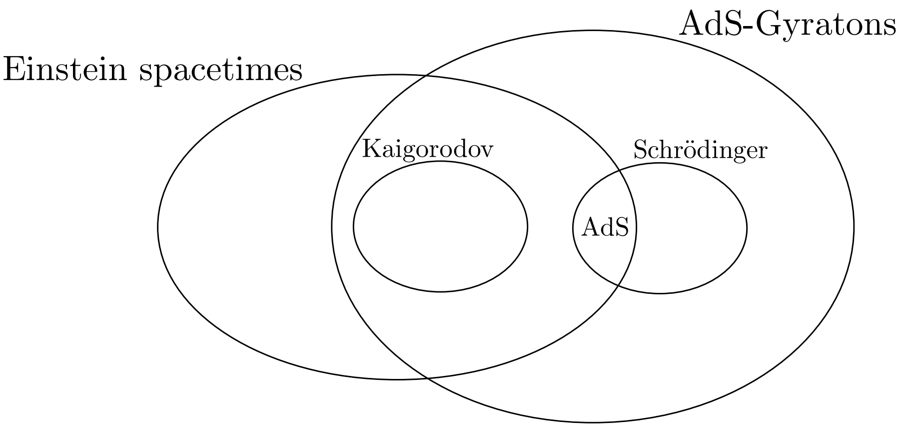


Figure 2.5: AdS-gyratons

Platonic plane waves: Spacetimes whose line element reads:

$$ds^2 = \frac{1}{\vec{x}^2}\left[2dt\left(du - \bar{U}\left(t,\vec{x}\right)dt + \bar{A}_i\left(t,\vec{x}\right)dx^i\right) + d\vec{x}^2\right] . \tag{2.3.21}$$

This name has been chosen because they are indeed plane-fronted Platonic waves in $D = 4$ (and their wavefronts are cylinders $\mathbb{R} \times \mathbb{S}^{d-1}$ in higher dimensions), as can be seen in the spherical coordinates with radial coordinate $r = |\vec{x}|$. The $D = 4$ dimensional Platonic plane waves form the only class of nonhomogeneous plane-fronted proper Platonic spacetimes with constant scalar curvature invariants, as will be explained in Section 2.4.2. However, they seem of little physical interest since none of them are Einstein manifolds.

2.4 Geometric properties of Platonic gravitational waves

2.4.1 Global properties: completeness and causality

Since global issues are investigated in the present Section, one should be more specific about the global structure of the spacetimes which will be considered. For the sake of simplicity, we will restrict our analysis to Platonic waves with:

(i) topology $\mathbb{R}^2 \times \Sigma$, where $\mathbb{R}^2$ corresponds to the domain of (u, t) in the Brinkmann coordinates,

(ii) conformal factor Ω and components $g_{\alpha\beta}$ of the spacetime metric that are regular functions of t and x^i,

(iii) geodesically complete wavefronts Σ endowed with the metric g_{ij},

(iv) conformally related Bargmann-Eisenhart waves such that their wavefronts Σ are endowed with a time-independent metric $\bar{g}_{ij}$ and are geodesically complete.

Physically, an important property of spacetimes is the absence of singularities, in the sense of geodesic completeness. Effectively, the geodesics of Platonic waves are described as trajectories for a dynamical system (1.3.1) defined in terms of the components of the metric $\bar{g}_{ij}$, the vector potential $\bar{A}_i$ and the effective potential $\bar{V}$. Due to the above simplifying assumptions, the only way for a geodesic to be incomplete in this restricted class of Platonic waves is that the corresponding dynamical trajectory goes to spatial infinity in a finite time.

Heuristically, one might expect that the radial behavior of the effective potential at spatial infinity controls the motion of observers at large distances, so that the behaviors of the conformal factor and scalar potential would control the geodesic completeness of Platonic waves. Indeed, these ideas can be converted into a Theorem, which is a perfect example of the utility of the ambient approach in the study of gravitational waves. Its proof is essentially a byproduct of Eisenhart-Lichnerowitz Theorem, *i.e* the geodesic completeness of Platonic waves follows from the completeness of the corresponding dynamical trajectories. For Bargmann-Eisenhart waves, this is an equivalence [111]. The distinction arises for proper Platonic waves (*i.e.* $\Omega \neq$const) because of the fact that finite time intervals $\Delta t = t_1 - t_0$ along a dynamical trajectory always correspond to finite affine-parameter intervals

$$\Delta\tau = \tau_1 - \tau_0 = \frac{1}{m}\int_{t_0}^{t_1} \Omega\big(t, x(t)\big)\, dt \tag{2.4.22}$$

along an ambient geodesic, since by assumption (ii) the conformal factor Ω is finite for any value of t and x^i. However, the converse is not necessarily true because if Ω tends to zero when $|t| \to \infty$, then $\Delta\tau$ may be finite even for infinite $|\Delta t|$.

In order to state our result, some definitions should be introduced. Let us denote by

$$\|x\| = \int_{x_0}^{x} \sqrt{\bar{g}_{ij}(x')\, dx'^i dx'^j} \tag{2.4.23}$$

the geodesic distance from the "origin" (chosen to be any given point) x_0 on Σ. "Spatial infinity" corresponds to the limit $\|x\| \to \infty$.

Definition 2.4.1 (Candela, Romero, Sánchez [112, 113]). *A function $f(t,x)$ on $\mathbb{R}\times\Sigma$ grows at most quadratically along finite times if for each $T > 0$ there exist some positive constants A_T and B_T such that*

$$f(x,t) \leqslant A_T\|x\|^2 + B_T \qquad \forall (t,x) \in [-T,T]\times\Sigma\,.$$

The function is said to grow subquadratically along finite times if the inequality is strict.

A Corollary [14] of the works [111, 112, 113] is the following fact:

Proposition 2.4.2 (Candela, Flores, Romero, Sánchez [111, 112, 113]). *Bargmann-Eisenhart waves obeying to conditions (i)-(iii) and with potential $\bar{U}(x,t)$ decreasing [i.e. $-\bar{U}(x,t)$ growing] at most quadratically at spatial infinity along finite times are geodesically complete.*

Therefore, by merely adapting the powerful results of [112, 113] (in particular Theorem 2) on the completeness of dynamical trajectories, one can show:

14. Their Corollary was not stated with the same degree of generality as formulated here, though the authors of [111, 112, 113] must be aware of this stronger formulation since it follows in a straightforward way from their many results.

Proposition 2.4.3. *Platonic waves obeying to conditions (i)-(iv) with:*

- *conformal factor* $\Omega(t,x)$,
- *minus the scalar potential* $-\bar{U}(x,t)$,
- *absolute value of the time derivative of the conformal factor* $|\partial_t\Omega(t,x)|$,
- *absolute value of the time derivative of the scalar potential* $|\partial_t\bar{U}(t,x)|$,

that grow at most quadratically at spatial infinity along finite times, are geodesically complete.

One should stress that the above bounds on the growths are with respect to the geodesic distance on Σ defined by the spatial metric $\bar{g}_{ij}$ (so *not* by the wavefront metric $g_{ij} = \Omega\bar{g}_{ij}$).

Proof: Ambient geodesics with $m = 0$ are effectively described as geodesics of the wavefronts Σ with respect to the metric g_{ij}. They are ensured to be complete by hypothesis (iii).

Ambient geodesics with $m \neq 0$ are effectively described as dynamical trajectories with respect to the action principle (1.3.4). Theorem 2 of [112, 113] applies because of hypotheses (i)-(iv) and ensures that they are complete if minus the effective potential $-\bar{V}$ and the absolute value of its time derivative $|\partial_t\bar{V}|$ grow at most quadratically along finite times. Indeed, the effective potential $\bar{V} = \bar{U} + \frac{1}{2}\frac{M^2}{m^2}\Omega$, defined by (1.3.11), decreases at most quadratically at finite times for all values of $M^2 \in \mathbb{R}$ because of the four hypotheses on the growing behavior. Similarly, $|\partial_t\bar{V}| \leqslant |\partial_t\bar{U}| + \frac{1}{2}\,|\frac{M^2}{m^2}|\,|\partial_t\Omega|$ grows at most quadratically at finite times. □

Application: Schrödinger spacetimes Sch_Z with $Z \geqslant 2$ are expected to be geodesically complete gravitational waves, as follows from the above Proposition. This remains obscure in the local Poincaré-like coordinates but becomes more manifest in the global "trapping" coordinates

$$ds^2 = \frac{1}{z^2}\left[2dt\left(du - \frac{1}{2}\Big(\cos^{2(Z-2)}(t)\, z^{2(1-Z)} + z^2 + \vec{y}^2\Big)dt + dz^2 + d\vec{y}^2\right)\right] \tag{2.4.24}$$

introduced in [114, 115] for this purpose. The Schrödinger spacetimes with $Z = 2$ were proved to be geodesically complete in [114, 115] but the case $Z > 2$ was left open. The domain $0 < z < \infty$ fulfills the assumptions (i)-(iii) for $Z > 2$ (this condition ensures the regularity of the scalar potential). The conformal factor and scalar potential satisfy the hypotheses of the Proposition 2.4.3 for $Z > 5/2$. Indeed, for all $Z > 1$, Ω and $\partial_t\Omega$ go to zero when z goes to ∞ and $-\bar{U} < 0$. Moreover, $|\partial_t\bar{U}| = \left|(Z-2)\cos^{2Z-5}(t)\sin(t)\, z^{2(1-Z)}\right|$ grows at most quadratically in z for $Z > 5/2$. Strictly speaking, the assumption (iv) is not satisfied because $\bar{g}_{ij} = \delta_{ij}$ is the flat metric and the half-space $0 < z < \infty$ is not geodesically complete since straight lines may cross the boundary $z = 0$. Nevertheless, this subtlety should not be a problem in regard of the geodesic completeness taking into account the known fact from [114, 115] that, for $Z \geqslant 2$, timelike and lightlike geodesics cannot reach $z = 0$ for a finite value of the affine parameter. Still, this fact prevents us from a full rigorous proof of the geodesic completeness for $Z > 2$.[15]

Another important global property of spacetimes is their causal structure. By definition, a Platonic wave is conformally related to a Bargmann-Eisenhart wave; thus, both share locally the same causal structure. Therefore, without loss of generality one may restrict the study of causal properties of the Platonic waves to the one of Bargmann-Eisenhart waves, being careful about the domain of definition of the conformal map. Platonic waves are causal spacetimes [116] but not more in general.[16] For instance, a celebrated result of Penrose is his proof [117] that exact plane waves are strongly causal but not globally hyperbolic (nor causally simple). As a byproduct of the ambient approach, the property of causal simplicity of Bargmann-Eisenhart waves was shown to be equivalent (modulo technical assumptions) to the existence of maximisers for the proper time between causally related events [88].

As geodesic completeness, the causal structure of Platonic waves is governed by the behavior of the potential at spatial infinity. Indeed, the following Theorem was shown [116] for Bargmann-Eisenhart waves $\mathbb{R}^2 \times \Sigma$ which are Coriolis-free and with time-independent geodesically-complete wavefronts Σ: if the potential decreases, at spatial infinity, with respect to the Riemannian distance on the wavefront (I) at most quadratically, then it is strongly causal, or (II) subquadratically, then it is globally hyperbolic. There is a wide class of relevant gravitational waves which satisfy the assumption (I) but not (II) and which are geodesically complete and strongly causal but not globally hyperbolic. Exact plane wave solutions and anti de Sitter spacetimes are the perfect example of such Platonic waves.

As one can see, the faster the potential decreases, the weaker is the causal structure of the Platonic wave. In fact, another result of [116] for these same generic classes of spacetimes is that: if the potential is nonpositive and decreases superquadratically (*i.e.* faster than $-\|x\|^2$) at spatial infinity, then it is not distinguishing (which is the weakest condition coming after mere causality). In any case, an important lesson to draw is that Platonic waves should be such that their scalar potential is bounded from below or at most decreases slowly at spatial infinity in order to have standard causality properties and no singularity.

Finally, because of the importance of black objects in contemporary general relativity, another important global issue is the existence of an event horizon. Partial answers are

15. The upper bound $Z > 5/2$ can be optimised till $Z > 2$ by adapting the Corollary 3 of [112, 113].
16. We refer to the Sections 4.1 of the review [95] for a useful reminder of the hierarchy of causality conditions in general relativity.

that Coriolis-free pp-waves cannot possess a horizon while some examples of Platonic waves do possess one [118]. However, such black waves are generated by somewhat exotic matter and it has been shown that a large class of Platonic waves with a regular horizon cannot be solutions of Einstein equations in vacuum or with null matter [119].

2.4.2 Curvature scalar invariants: classification

Curvature scalar invariants (*i.e.* scalars built as polynomials formed from the Riemann tensor and its Koszul connections) constitute a powerful tool in the equivalence problem, that is the task to determine if two given metrics are locally isomorphic or not. As such, Riemannian manifolds are entirely determined by their curvature scalar invariants [120] and one is then able to tell if two Riemannian manifolds are isomorphic by systematically comparing their respective curvature scalar invariants. For Lorentzian spacetimes though, this Theorem does not hold and there exists a nontrivial class of spacetimes which are not uniquely characterised by their invariants so that more elaborate procedures such as the Cartan-Karlhede algorithm are needed in order to solve the equivalence problem. In four dimensions, this special class of spacetimes is identified with the one of degenerate-Kundt metrics [121], introduced in Section 2.2.4, so that nonequivalent degenerate Kundt metrics can share identical invariants. Although it stays true that degenerate-Kundt spacetimes are not determined by their scalar invariants in higher dimensions [122], it remains to be proved that they are the only higher dimensional spacetimes enjoying this property. We established earlier that Platonic waves are degenerate-Kundt; therefore, we formulate the following:

Proposition 2.4.4. *Platonic waves are not determined by their scalar curvature invariants.*

The very existence of a class of spacetimes not being characterised by their invariants opens the possibility of Lorentzian manifolds having vanishing curvature scalar invariants (called VSI spacetimes in the following) without necessarily being flat. As is obvious from the previously stated Theorem, the only Riemannian VSI manifolds are flat. By definition, curved Lorentzian VSI manifolds are not determined by their scalar curvature invariants and, furthermore, it can be shown that they belong to the degenerate-Kundt class in any dimension [123]. The authors of [124] showed that in arbitrary dimension a spacetime is VSI if and only if it belongs to the Kundt class, *i.e.* admits a geodesic nonexpanding, shear-free and twist-free null vector field ξ, and the Riemann tensor is of type III (or more special) relative to ξ. The second condition involves the notion of the boost order of a tensor, which we define, following the terminology introduced in [125, 126] (*cf.* [127] for a pedagogical review), as the difference between the number of "+" and "−" in the components of a covariant tensor (concretely, all down indices) written in an adapted frame. The condition prescribing that the Riemann tensor of a VSI spacetime must be of type III relative to ξ is equivalent to have a Riemann tensor with strictly negative boost order when computed in the adapted frame.

Concretely, the condition that the boost order of the Riemann tensor is strictly negative amounts to the set of equations below:

Boost order	Riemann component
2	$R_{+i\|+j} = 0$
1	$R_{+-\|+i} = R_{+i\|jk} = 0$
0	$R_{+-\|+-} = R_{+-\|ij} = R_{+i\|-j} = R_{ij\|kl} = 0$

We now focus on the VSI spacetimes among the Platonic waves and prove the following Lemma:

Lemma 2.4.5. *A Bargmann-Eisenhart wave is VSI if and only if it is a pp-wave.*

Proof: Platonic waves belong to the Kundt class so the only remaining condition to satisfy is that the boost order of the Riemann tensor is negative.

The existence of a congruence of parallel rays implies that $R_{ab|cd}\xi^d = 0$ and thus we have, in the kinematic frame, $R_{ab|c+} = 0$ since ξ is nowhere vanishing. Therefore the first six conditions are automatically satisfied for a Bargmann-Eisenhart wave. The last condition is equivalent to be plane fronted. □

The extension of this result to Platonic waves is rendered quite simple by the useful result of [128] stating that if a VSI spacetime admits a null (or timelike) Killing vector field ξ, then ξ is necessarily parallel. Therefore the class of VSI Platonic waves reduces to the one of VSI Bargmann-Eisenhart spacetimes and we have the following Proposition:

Proposition 2.4.6. *A Platonic wave is VSI if and only if it is a pp-wave.*

One way to heuristically interpret this result is to consider that the VSI property of a Platonic wave descends to the wavefront, which being Riemannian, must necessarily be flat.

We now consider the natural extension of the VSI class that is spacetimes possessing *constant* curvature scalar invariants (CSI). For Riemannian manifolds, the class of CSI metrics reduces to (locally) homogeneous manifolds [129]. The Lorentzian case is again richer as, in four dimensions, the CSI class is composed of all (locally) homogeneous manifolds as well as a subset of the degenerate Kundt spacetimes dubbed *degenerate-CSI$_K$* metrics [130]. Degenerate-CSI$_K$ are Kundt spacetimes for which there exists a frame such that all curvature tensors (that is the Riemann tensor and all its Koszul connections) have vanishing positive boost weight components and constant boost weight zero components. In higher dimensions, the situation is less clear than in the VSI case as it is not yet known if the class of (locally) homogeneous spacetimes together with the class of degenerate-CSI$_K$ spacetimes exhaust the CSI class when $D > 4$. For this reason, we will focus in the sequel on the $D = 4$ case.

We again start with the Bargmann-Eisenhart case. Actually, this question has already been addressed in [131] (where Bargmann-Eisenhart spacetimes are denoted CCNV) and the following Proposition has been established:

Proposition 2.4.7 (McNutt, Coley, Pelavas [131]). *A four-dimensional Bargmann-Eisenhart wave is CSI if and only if its wavefront is locally homogeneous.*

Again, we note that the CSI property seems to befall to the wavefront. There are three types of 2-dimensional locally homogeneous Riemannian spaces, respectively locally isometric to: the sphere $\mathbb{S}^2$, the Euclidean plane $\mathbb{E}^2$ and the hyperbolic plane $\mathbb{H}^2$. The general expression of a four-dimensional CSI Bargmann-Eisenhart spacetime, in Brinkmann coordinates then reads:

$$ds^2 = 2\,dt\left(du - \bar{U}(t,\vec{x})dt + \bar{A}_i(t,\vec{x})\,dx^i\right) + d\ell^2 \tag{2.4.25}$$

where the wavefront line element takes the form $d\ell^2 = dx^2 + \frac{1}{\lambda^2}\sin^2(\lambda x)\,dy^2$ where $\mathbb{S}^2$: $\lambda^2 > 0$, $\mathbb{E}^2$: $\lambda^2 = 0$ and $\mathbb{H}^2$: $\lambda^2 < 0$. Obviously, the Euclidean case corresponds to a pp-wave and the spacetime is then VSI. In order to address the Platonic case, we will rely on the classification of four dimensional degenerate-CSI$_K$ metrics proposed in [130] and prove the following Proposition:

Proposition 2.4.8. *A four-dimensional Platonic wave is CSI if and only if it belongs to one of the following classes:*

- *locally homogeneous*
- *CSI Bargmann-Eisenhart*
- *AdS-gyraton*
- *Platonic plane wave.*

Proof: As stated earlier, four-dimensional CSI spacetimes consist of all locally homogeneous or degenerate-CSI$_K$ spacetimes [17]. We now focus on Platonic waves

17. Note that these two classes intersect, see *e.g.* footnote 14 in [132].

belonging to the degenerate-CSI$_K$ class and make use of the classification of four-dimensional degenerate-CSI$_K$ displayed in [130]. More technically, the authors of [130] wrote, for each class of locally homogeneous wavefront (*i.e.* $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{H}^2$) the two-dimensional one-forms $\tilde{A}_i^{(1)}$ allowing the construction of a degenerate-CSI$_K$ spacetime. Our task is then to require that the obtained line element matches the form of Platonic waves seen as degenerate-Kundt metrics (*cf.* Section 2.2.4) for some function $\Omega(t, x)$. This requirement is quite drastic as, besides the CSI Bargmann-Eisenhart, only two classes of proper Platonic waves remain, namely AdS-gyratons and Platonic plane waves. $\square$

We note that no nonhomogeneous spherical wavefront proper Platonic waves are CSI. The figure 2.6 provides a summary of the Platonic CSI spacetimes.

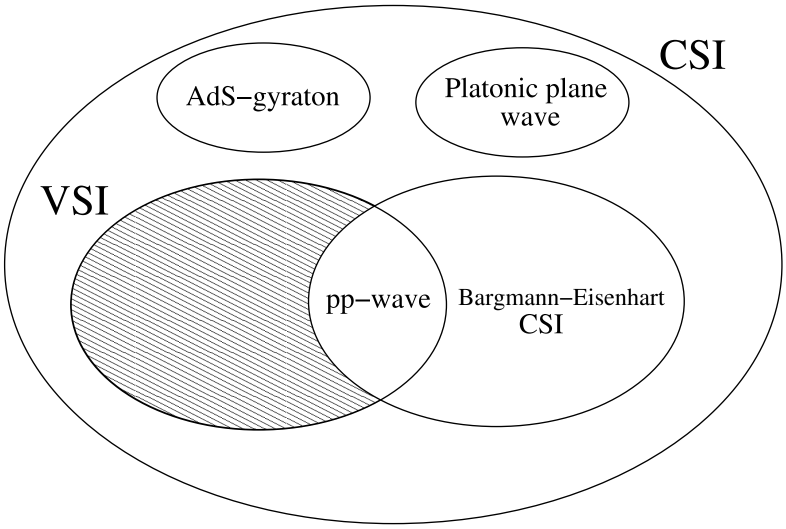


Figure 2.6: Four-dimensional Platonic CSI spacetimes: Note that the set VSI$\setminus$pp-wave is empty.

The study of CSI spacetimes is partly motivated by the physically relevant notion of "universality" which designates the property enjoyed by spacetimes which are vacuum solutions of any theory of quantum gravity (in the sense of effective field theory, *e.g.* the string theory low-energy effective action). A more precise definition [108] distinguishes between weakly (and strongly) universal spacetimes to designate spacetimes for which any conserved symmetric tensor of rank two constructed from the metric, the Riemann tensor and its Koszul connections is a constant multiple of the metric (vanishes). The link between the universality and CSI properties has been highlighted in [133], where it was shown that any universal four-dimensional spacetime must be CSI. However, there is still no crisp result allowing us to discriminate which CSI spacetimes are universal.

A conjectured candidate for a subset of universal CSI spacetimes are the so-called CSI_Λ spacetimes [134] whose invariants constructed from the traceless Ricci tensor, Weyl tensor and their Koszul connections vanish. The work [134] displays a classification of four-dimensional CSI_Λ spacetimes which relies on the one proposed in [130]. Then, by similar arguments as the one used in the proof of Proposition 2.4.8, we establish the following fact:

Proposition 2.4.9. *CSI_Λ Platonic waves are either pp-waves or AdS-gyratons.*

Indeed, this class contains the two classes of universal Platonic waves already known in the literature: Coriolis-free pp-waves have been shown to be strongly universal in [135] while Siklos waves are known to be weakly universal [108].

Chapter 3

Intrinsic geometric structures: relativistic vs nonrelativistic

In the present Chapter, various geometric structures are introduced in a purely intrinsic manner, *i.e.* without reference to the notion of embedding nor to Cartan geometry, which will be examined further (respectively in Chapters 4 and 6). The content of this Chapter consists mostly in a review of the existing literature, with the exception of the last Section which proposes some new notions of parallelism.

After a briew review of standard definitions and properties regarding relativistic structures (Section 3.1), we switch to the investigation of nonrelativistic structures (Section 3.2) by emphasising their points of divergence with their relativistic avatars. We first review nonrelativistic metric structures in Section 3.2.1. These are characterised by a manifold endowed with a degenerate contravariant metric whose radical (*cf.* Definition A.1.2) is spanned by a 1-form, dubbed the *absolute clock*. The role played by fields of observers in nonrelativistic physics will be discussed at length as well as related objects. We then discuss two restrictions that can be imposed on the absolute clock 1-form, namely when the 1-form is closed (*Augustinian* structure) or satisfy the Frobenius Criterion (*Aristotelian* structure). In Section 3.2.2, we discuss the possibility to endow nonrelativistic metric structures with a notion of parallelism, in the guise of a Koszul connection [1]. We first focus on torsionfree Koszul connections compatible with the underlying metric structure, thus restricting the scope of the analysis to Augustinian structures. We thus review the notions of *Galilean* and *Newtonian* connections, with particular attention given to the equivalence problem (*i.e.* the search for necessary structures in order to uniquely determine a given

1. We will prefer the denomination "Koszul connection" to the more widespread designations of "covariant derivative" or "affine connection" in order to avoid confusion with the different meanings of these terms in the mathematical literature.

connection). Apart from the standard characterisation of Newtonian connections in terms of equivalence classes of field of observers and 1-forms, this approach will lead us to review the less standard *Lagrangian* formulation of Newtonian manifolds. Section 3.2.3 will conclude this Chapter by examining two propositions of connections endowing Aristotelian structures, dubbed respectively *Horizontal* and *Platonic* connections. The definition of these two notions of connection will rely on the generalisation to the Aristotelian case of the "standard" and Lagrangian formulations of Newtonian connections, respectively.

Minuscule Greek indices $\mu, \nu, ...$ will denote spacetime indices taking $d{+}1$ values $(0, 1, 2, \dots, d)$ while minuscule latin indices as $i, j, ...$ will denote spatial indices taking d values $(1, 2, \dots, d)$.

3.1 Relativistic structures

We start by reviewing some standard material about relativistic structures in order to draw comparison with nonrelativistic ones and fix some terminology.

Definition 3.1.1 (Riemannian structure)**.** *A Riemannian structure designates a manifold endowed with a positive-definite metric.*

Although this definition restricts to the case of signature $(+, \dots, +)$, a similar one can be given in the pseudo-Riemannian case:

Definition 3.1.2 (Lorentzian structure)**.** *A Lorentzian structure consists in a manifold endowed with a non-degenerate metric of signature* $(-, +, \dots, +)$.

These "relativistic" structures are therefore characterised by a metric structure but, as such, are not endowed with a notion of parallel transport. This supplementary notion of parallelism can be implemented under the features of a Koszul connection (*cf.* Definition A.9.1) compatible with the metric structure. The term "compatible" is here to be understood in the sense of Definition A.9.4. We are thus led to define:

Definition 3.1.3 (Riemannian/Lorentzian manifold)**.** *A Riemannian (Lorentzian) manifold consists in a Riemannian (Lorentzian) structure supplemented with a compatible Koszul connection.*

We will retain this convention in the following and use the word "structure" in order to designate a manifold endowed with a metric-like structure while keeping the term "manifold" for cases where a compatible Koszul connection is added. However, the distinction drawn here is only relevant when the Koszul connection has torsion (*cf.* Definition A.9.2), due to the following Theorem and its Corollary:

Theorem 3.1.4 (Space of metric compatible connections)**.** *Let g be a (pseudo)-Riemannian metric on a manifold $\mathscr{M}$. The set of all Koszul connections compatible with the metric g forms a vector space isomorphic to the vector space of all possible torsions.*

Theorem 3.1.4 is a consequence of Cartan's Lemma and can be restated in our terminology by saying that a given Riemannian/Lorentzian structure supplemented with a vector field valued 2-form (the torsion) defines uniquely a Riemannian/Lorentzian manifold. Furthermore, given a particular metric structure, there is no restriction on the possible torsion which can span the whole vector space of vector field valued 2-forms. As we will see, these features are characteristic of the relativistic structures and will be one of the main points of discrepancy with nonrelativistic structures. The following Corollary is immediate:

Theorem 3.1.5 (Fundamental Theorem of (pseudo)-Riemannian geometry)**.** *There is a unique torsionfree Koszul connection compatible with a given (pseudo)-Riemannian metric.*

This torsionfree Koszul connection is called the Levi-Civita connection and plays the role of the zero of the vector space corresponding to all possible Koszul connections compatible with the metric structure. This should be distinguished from the case where no metric structure is involved: in that case, the space of all Koszul connections on a manifold possesses a structure of *affine space*, with associated vector space the space of 1-contravariant, 2-covariant tensor fields $\Gamma\left(T\mathscr{M}\otimes\vee^2 T^*\mathscr{M}\right)$. This translates the well-known fact that the difference between two Koszul connections on the same manifold is a tensor $\big($an element of the vector space $\Gamma\left(T\mathscr{M}\otimes\vee^2 T^*\mathscr{M}\right)\big)$ although a Koszul connection is not. In holonomic coordinates, if one writes $\nabla_\mu Y^\lambda = \partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu} Y^\nu$, then the components $\Gamma^\lambda_{\mu\nu}$ defining the Levi-Civita connection are equal to the usual Christoffel symbol:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right). \tag{3.1.1}$$

We stress that Theorem 3.1.5 involves no restriction on the metric structure, so that any Riemannian/Lorentzian structure induces a unique torsionfree Koszul connection. As we will see, this property is lost when one deals with degenerate metric structures.

Definition 3.1.6 (Lorentzian basis)**.** *Let $(\mathscr{M}, g)$ be a $(d+1)$-dimensional Lorentzian structure with non-degenerate covariant metric g. A Lorentzian basis of the tangent space $T_x\mathscr{M}$ at a point $x \in \mathscr{M}$ is an ordered basis $B_x = \{e_{0x}, \dots, e_{dx}\}$ which is orthonormal with respect to g_x.*

The basis vectors e_{ax}, with $a \in \{0, \cdots, d\}$ thus satisfy the condition $g_x\left(e_{ax}, e_{bx}\right) = \eta_{ab}$, with η_{ab} the Minkowski metric. The denomination Lorentzian is justified by the following Proposition:

Proposition 3.1.7. *At each point $x \in \mathscr{M}$, the set of endomorphisms of $T_x\mathscr{M}$ mapping each Lorentzian basis into another one forms a group isomorphic to the Lorentz group.*

Conformal relativistic structures

Definition 3.1.8 (Conformal class of metrics). *A conformal class of metrics is an equivalence class, denoted $[g]$, in which two metrics g and $\tilde{g} \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ are said to be equivalent if there exists a positive function $\Omega \in C^\infty(\mathscr{M})$ such that $\tilde{g} = \Omega\, g$.*

Regarding parallelism, there is no Levi-Civita connection compatible with a given conformal class, as the one preserving a given representative g of the class will not preserve a metric $\tilde{g} = \Omega\, g$ related to the first one by a (non-constant) conformal factor Ω. Each metric thus induces its own Levi-Civita connection and the best that can be done is to specify the transformation relations between Christoffel symbols. Explicitly, if we denote by $\tilde{\nabla}$ and ∇ the respective Levi-Civita connections associated to the conformally related metrics $\tilde{g}$ and g with conformal factor Ω such that $\tilde{g} = \Omega\, g$, thus $\tilde{\nabla}\tilde{g} = 0$ and $\nabla g = 0$ so that the Christoffel symbols for $\tilde{\nabla}$ read:

$$\begin{aligned} \tilde{\Gamma}^\lambda_{\mu\nu} &= \frac{1}{2}\tilde{g}^{\lambda\rho}\left(\partial_\mu \tilde{g}_{\rho\nu} + \partial_\nu \tilde{g}_{\rho\mu} - \partial_\rho \tilde{g}_{\mu\nu}\right) \\ &= \Gamma^\lambda_{\mu\nu} + \frac{1}{2}\left(\delta^\lambda_\mu \partial_\nu \ln\Omega + \delta^\lambda_\nu \partial_\mu \ln\Omega - g^{\lambda\rho} g_{\mu\nu}\, \partial_\rho \ln\Omega\right) \end{aligned} \tag{3.1.2}$$

with $\Gamma^\lambda_{\mu\nu}$ the Christoffel symbols of ∇ given by eq.(3.1.1). One can make use of eq. (3.1.2) in order to compute $\tilde{\nabla} g = -d(\ln\Omega)\, g$. The metric g is therefore said recurrent (*cf. e.g.* [136]) with respect to $\tilde{\nabla}$, with recurrence 1-form $-d(\ln\Omega)$.

Definition 3.1.9 (Weyl structure). *A conformal class $[g]$ of metrics supplemented with a map $F : [g] \to \Omega^1(\mathscr{M})$ satisfying $F\left(e^\lambda g\right) = F(g) - d\lambda$, $\forall\ \lambda \in C^\infty(\mathscr{M})$ and $g \in [g]$ is called a Weyl structure.*

A Weyl structure $([g], F)$ can be seen as an equivalence class $[(g, \omega)]$ with $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ and $\omega \in \Omega^1(\mathscr{M})$ in which two couples (g, ω) and (g', ω') are said equivalent if they satisfy the *Eichtransformation*[2]:

$$g' = e^\lambda g \quad \text{and} \quad \omega' = \omega - d\lambda \tag{3.1.3}$$

2. The term Eichtransformation (german for Gauge transformation) and the associated transformation were first written by H.Weyl in his book *Gravitation und Elektrizität* (1918) in an attempt to describe gravitation and electromagnetism in an unifying geometric framework.

for some $\lambda \in C^\infty(\mathscr{M})$. To a given Weyl structure can be associated a torsionfree Koszul connection ∇, called the Weyl connection, whose components in a holonomic basis are given by:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) + \frac{1}{2}\left(\delta^\lambda_\mu \omega_\nu + \delta^\lambda_\nu \omega_\mu - g^{\lambda\rho} g_{\mu\nu}\,\omega_\rho\right). \quad (3.1.4)$$

The preceding expression can be recast in the form (3.1.1) with the partial derivative operators replaced by "Maxwell covariant derivatives": $\partial_\mu \to D_\mu = \partial_\mu + \omega_\mu$. This makes the coefficients (3.1.4) explicitly invariant under an Eichtransformation (3.1.3). However, it should be noted that the Weyl connection ∇ corresponding to coefficients (3.1.4) does not preserve any of the (conformally related) metrics $g \in [g]$. Rather, the metrics g are recurrent with respect to ∇, *i.e.* they satisfy $\nabla_\mu g_{\alpha\beta} = -\omega_\mu g_{\alpha\beta}$, with recurrence 1-form $-\omega$.

Definition 3.1.10 (Integrable Weyl structure)**.** *A Weyl structure $[(g, \omega)]$ in which one of the representative 1-forms ω is closed is called integrable.*

Proposition 3.1.11 (*cf. e.g.* [137])**.** *Integrable Weyl structures are in one-to-one correspondence with Lorentzian structures.*

Proof: In an integrable Weyl structure, since one of the 1-forms is closed, all of them are, thanks to the transformation law (3.1.3), so that all the 1-forms of the class can locally be considered as exact. In particular, one of the 1-forms vanishes and thus defines a privileged metric, denoted g. Conversely, given a Lorentzian structure with metric g, one can construct the couple $(g, 0)$ from which all the other representatives can be deduced by performing (3.1.3). □

Note that, in the torsionfree case, the Weyl connection ∇ is the Levi-Civita connection associated to the privileged metric g (since the coefficients in (3.1.4) become equal to those of (3.1.1) whenever ω vanishes), so that g is the only metric of the integrable Weyl class to be preserved by ∇.

3.2 Nonrelativistic structures

3.2.1 Nonrelativistic metric structures

As hinted in the previous Section, a crucial ingredient of nonrelativistic structures is the presence of a contravariant (resp. covariant) degenerate metric [3] [11, 12, 14], whose radical

3. Throughout this work, the term "metric" will be used in a slightly broader sense than the customary one in the physics literature. Namely, we will employ the term to designate a covariant or contravariant bilinear form being either degenerate or non-degenerate.

is spanned by a given 1-form (resp. vector field), which must be separately specified. More explicitly, we can define the nonrelativistic analogue of a Riemannian structure, dubbed a Leibnizian structure as follows:

Definition 3.2.1 (Leibnizian structure [18]). *A Leibnizian structure comprises the following three elements:*

- *a manifold* $\mathscr{M}$
- *a nowhere vanishing 1-form* $\psi \in \Omega^1(\mathscr{M})$
- *a positive semi-definite contravariant metric* $h \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ *with radical* $\text{Rad } h = \text{Span } \psi$.

In holonomic coordinates, the condition that $\text{Rad } h = \text{Span } \psi$ reads $h^{\mu\nu}\psi_\nu = 0$. If one insists in dealing with a non-degenerate metric, an alternative definition of a Leibnizian structure can be formulated as follows:

Definition 3.2.2 (Leibnizian structure [92]). *A Leibnizian structure consists of a triplet composed of the following elements:*

- *a manifold* $\mathscr{M}$
- *a nowhere vanishing 1-form* $\psi \in \Omega^1(\mathscr{M})$
- *a positive-definite covariant metric* γ *acting on*[4] $\text{Ker } \psi$, $\gamma \in \Gamma\left(\vee^2 (\text{Ker } \psi)^*\right)$.

Proposition 3.2.3. *Definitions 3.2.1 and 3.2.2 are equivalent.*

Proof: The proof follows straightforwardly from Proposition A.1.5. □

We will call ψ an *absolute clock* and γ a collection of *rulers*. The absolute clock allows to distinguish between *timelike* tangent vectors $X_x \in T_x\mathscr{M}$ for which $\psi_x(X_x) \neq 0$ from *spacelike* tangent vectors $Y_x \in T_x\mathscr{M}$ satisfying $\psi_x(Y_x) = 0$.

Definition 3.2.4 (Observer). *An observer is a timelike curve* $n : I \subseteq \mathbb{R} \to \mathscr{M} : s \mapsto n(s)$ *normalised such that the tangent vector* $N_{n(s)} \in T_{n(s)}\mathscr{M}$ *(defined*[5] *as* $N_{n(s)} \equiv n_* D_s$*) satisfies:*

$$\psi_{n(s)}\left(N_{n(s)}\right) = 1, \ \forall\, s \in I. \tag{3.2.5}$$

This notion can be generalised to define fields whose integral curves are observers:

Definition 3.2.5 (Field of observers). *A field of observers is a vector field* $N \in \Gamma(T\mathscr{M})$ *such that* $\psi(N) = 1$. *The space of all fields of observers on* $\mathscr{M}$ *is denoted* $FO(\mathscr{M})$.

4. At each point $x \in \mathscr{M}$, $\text{Ker } \psi_x$ stands for the subspace of $T_x\mathscr{M}$ spanned by vectors annihilated by ψ_x and $\text{Ker } \psi$ must thus be understood as the subbundle of $T\mathscr{M}$ of vector fields annihilated by the 1-form ψ.

5. The vector $D_s \in T\mathbb{R}_s$ is defined by its action on functions $f \in C^\infty(\mathbb{R})$ as $D_s[f] = \left.\frac{\partial f}{\partial t}\right|_{t=s}$.

Definition 3.2.6 (Proper time)**.** *Let $\mathscr{C}$ be a 1-dimensional immersed submanifold of $\mathscr{M}$ defined by the injective immersion $i : \mathscr{C} \to \mathscr{M}$. We will call proper time a function $\tau \in C^{\infty}(\mathscr{C})$ satisfying $d\tau = i^{*}\psi$.*

The fact that the submanifold $\mathscr{C}$ is of dimension 1 ensures that the 1-form $i^{*}\psi$ is closed, so that locally there always exists a function τ such that $d\tau = i^{*}\psi$. Obviously, this condition only defines the proper time up to a constant.

Definition 3.2.7 (Proper time parameterisation)**.** *Let $\mathscr{C}$ be a 1-dimensional immersed submanifold of $\mathscr{M}$ defined by the injective immersion $i : \mathscr{C} \to \mathscr{M}$ and let $x : I \subseteq \mathbb{R} \to \mathscr{C} : s \mapsto x(s)$ be a curve on $\mathscr{C}$. The curve x is said to be parameterised by the proper time τ if*

$$\tau \circ x(s) = s + a, \ \forall s \in I \tag{3.2.6}$$

with $a \in \mathbb{R}$ a constant.

Proposition 3.2.8. *Let $\mathscr{C}$ be a 1-dimensional immersed submanifold of $\mathscr{M}$ defined by the injective immersion $i : \mathscr{C} \to \mathscr{M}$ and let $x : I \subseteq \mathbb{R} \to \mathscr{C} : s \mapsto x(s)$ be a curve on $\mathscr{C}$. The curve x is parameterised by the proper time τ if and only if the curve $n : I \to \mathscr{M}$ defined as $n \equiv i \circ x$ is an observer.*

Proof: We start from eq.(3.2.5) and show the following string of equivalences:

$$\begin{aligned}
\psi_{n(s)}\left(N_{n(s)}\right) = 1, \ \forall s \in I \ &\Leftrightarrow \ \psi_{n(s)}\left(n_{*}D_{s}\right) = 1, \ \forall s \in I \text{ (Definition of a tangent vector)} \\
&\Leftrightarrow \ \psi_{n(s)}\left((i \circ x)_{*} D_{s}\right) = 1, \ \forall s \in I \text{ (Definition of } n\text{)} \\
&\Leftrightarrow \ \left(i^{*}\psi_{n(s)}\right)_{x(s)}\left(x_{*}D_{s}\right) = 1, \ \forall s \in I \text{ (eq.(A.2.5))} \\
&\Leftrightarrow \ d\tau_{x(s)}\left(x_{*}D_{s}\right) = 1, \ \forall s \in I \text{ (Definition 3.2.6)} \\
&\Leftrightarrow \ x_{*}D_{s}[\tau] = 1, \ \forall s \in I \text{ (Action of a differential on a vector)} \\
&\Leftrightarrow \ D_{s}[\tau \circ x] = 1, \ \forall s \in I \text{ (eq.(A.2.3))} \\
&\Leftrightarrow \ \left.\frac{\partial\, \tau \circ x(l)}{\partial l}\right|_{l=s} = 1, \ \forall s \in I \text{ (Definition of } D_{s}\text{)} \\
&\Leftrightarrow \ \exists a \in \mathbb{R} / \tau \circ x(s) = s + a, \ \forall s \in I.
\end{aligned}$$

□

Definition 3.2.9 (Spacelike projection of vector fields [92])**.** *Let $N \in FO(\mathscr{M})$ be a field of observers. The field of endomorphisms $P^{N} : \Gamma(T\mathscr{M}) \to \operatorname{Ker}\psi$ defined as*

$$P^{N}(X) = X - \psi(X)\, N \tag{3.2.7}$$

where X is any vector field, is called the spacelike projector of vector fields along N.

The transpose of a spacelike projector can be defined (*cf.* Definition (A.1.6)) as the field of endomorphisms[6] $\bar{P}^N : \Omega^1(\mathscr{M}) \to \text{Ann } N$ defined as $\bar{P}^N(\alpha) = \alpha - \alpha(N)\psi$, with $\alpha \in \Omega^1(\mathscr{M})$.

Given two fields of observers N' and N, their difference $S = N' - N$ belongs to the kernel of the absolute clock ($\psi(S) = 0$) so that $S \in \Gamma(T\mathscr{M})$ is not a field of observer. This observation prevents the space $FO(\mathscr{M})$ of all fields of observers $FO(\mathscr{M})$ from being a vector space. However, $FO(\mathscr{M})$ possesses a natural structure of *affine* space [92] with associated vector space Ker ψ. Since any element $X \in \text{Ker } \psi$ can be expressed using the Leibnizian contravariant metric h as $X \equiv h(\chi)$ for some 1-form $\chi \in \Omega^1(\mathscr{M})$, the next Proposition follows straightforwardly:

Proposition 3.2.10 (Milne boost [138, 139])**.** *Let N and $N' \in FO(\mathscr{M})$ be two fields of observers on $\mathscr{M}$. Then there exists a 1-form $\chi \in \Omega^1(\mathscr{M})$ such that $N' = N + h(\chi)$. The fields of observers N and N' are said to be related by a Milne boost parameterised by the 1-form χ.*

In components, the previous Proposition can be restated, assuming $\psi_\mu N^\mu = 1$ as

$$\psi_\mu N'^\mu = 1 \quad \Longleftrightarrow \quad \exists\, \chi_\nu \,/\, N'^\mu = N^\mu + h^{\mu\nu}\chi_\nu.$$

The 1-form χ only appears through the combination $h(\chi)$ and thus can be taken spacelike, *i.e.* χ may be everywhere replaced by $\bar{P}^N(\chi)$ for some arbitrary N. Milne boosts are sometimes referred to as local Galilean boosts, denomination that will be justified in Proposition 3.2.14. A spacelike 1-form χ can be thought of as the local relative speed between two fields of observers.

Fields of observers are bestowed upon a greater importance in nonrelativistic physics in comparison with the relativistic case, since a great deal of structures can only be defined by making use of a particular N (thus in a non-canonical way). Indeed, since the contravariant metric h of a Leibnizian structure is degenerate, there is no natural covariant metric defined on the whole tangent bundle $T\mathscr{M}$ (not just on Ker ψ). However, the gift of a field of observers N allows to uniquely define a (degenerate) covariant bilinear form $\overset{N}{\gamma}$ transverse to N as:

Definition 3.2.11 (Transverse metric)**.** *Let $\mathscr{L}(\mathscr{M}, \psi, \gamma)$ be a Leibnizian structure and $N \in FO(\mathscr{M})$ a field of observers on $\mathscr{M}$. The transverse metric $\overset{N}{\gamma} \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ is*

6. At each point $x \in \mathscr{M}$, Ann N_x stands for the annihilator of Span N_x in $T^*_x\mathscr{M}$ (*cf.* Definition A.1.1) and Ann N is thus to be understood as the subbundle of $T^*\mathscr{M}$ spanned by 1-forms annihilating the field of observers N.

defined by its action on vector fields $X, Y \in \Gamma(TM)$ as

$$\overset{N}{\gamma}(X,Y) = \gamma\left(P^N(X), P^N(Y)\right) \tag{3.2.8}$$

where $\gamma \in \Gamma\left(\vee^2 (\text{Ker } \psi)^*\right)$ is the spatial metric of the Leibnizian structure and P^N stands for the spacelike projector associated to N (cf. Definition 3.2.9).

The right-hand side of eq.(3.2.8) is well-defined since the image of a spacelike projection lies in Ker ψ. The epithet "transverse" is justified by the fact that $N \in \text{Rad}\,\overset{N}{\gamma}$. Furthermore, it is easy to show that the contraction of $\overset{N}{\gamma}$ with the contravariant metric h satisfies the relation: $h\left(\overset{N}{\gamma}(X)\right) = P^N(X), \forall\, X \in \Gamma(T\mathscr{M})$. In components, we thus have the two relations:

$$\begin{cases} \overset{N}{\gamma}_{\mu\nu}\, N^\nu = 0 \\ \overset{N}{\gamma}_{\nu\lambda}\, h^{\lambda\mu} = \delta^\mu_\nu - N^\mu \psi_\nu. \end{cases} \tag{3.2.9}$$

In fact, these two conditions completely determine $\overset{N}{\gamma}$, as expressed by the following Proposition:

Proposition 3.2.12 (See *e.g.* [18], Section 4)**.** *Let $\mathscr{L}(\mathscr{M}, \psi, \gamma)$ be a Leibnizian structure and $N \in FO(\mathscr{M})$ a field of observers on $\mathscr{M}$. There is a unique covariant bilinear form $\overset{N}{\gamma} \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ satisfying the conditions* (3.2.9).

As suggested by the superscript, the covariant metric $\overset{N}{\gamma}$ depends on the choice of field of observers N. More precisely, it can be shown that under a change of field of observers $N \to N'$ via the Milne boost parameterised by the 1-form $\chi \in \Omega^1(\mathscr{M})$, the metric $\overset{N}{\gamma}$ varies as

$$\overset{N}{\gamma}_{\mu\nu} \to \overset{N}{\gamma}_{\mu\nu} + \Big(2\,\chi(N) + h(\chi,\chi)\Big)\psi_\mu\psi_\nu - 2\,\chi_{(\mu}\psi_{\nu)}. \tag{3.2.10}$$

Definition 3.2.13 (Galilean basis)**.** *Let $\mathscr{L}(\mathscr{M}, \psi, \gamma)$ be a Leibnizian structure. A Galilean basis of the tangent space $T_x\mathscr{M}$ at a point $x \in \mathscr{M}$ is an ordered basis $B_x = \{N_x, e_{1x}, \ldots, e_{dx}\}$ with N_x the tangent vector of an observer and $\{e_{1x}, \ldots, e_{dx}\}$ a basis of* Ker ψ_x *which is orthonormal with respect to γ_x.*

Explicitly, the basis $B_x = \{N_x, e_{1x}, \ldots, e_{dx}\}$ must satisfy the conditions:

1. $\psi_x(N_x) = 1$
2. $\psi_x(e_{ix}) = 0\,, \forall\, i \in \{1, \ldots, d\}$
3. $\gamma_x(e_{ix}, e_{jx}) = \delta_{ij}\,, \forall\, i, j \in \{1, \ldots, d\}$.

A basis of $T^*_x\mathscr{M}$ dual to $B_x = \{N_x, e_{ix}\}$ is given by $B^*_x \equiv \left\{\psi_x, \theta^i_x\right\}$, where the d one-forms θ^i_x satisfy the requirement: $\theta^i_x\left(e_{jx}\right) = \delta^i_j$.

The reference to Galilei in Definition 3.2.13 is justified by the following Proposition:

Proposition 3.2.14 (See [92], Section 2.C)**.** *At each point $x \in \mathscr{M}$, the set of endomorphisms of $T_x\mathscr{M}$ mapping each Galilean basis into another one forms a group isomorphic to the homogeneous Galilei group* Gal_0.

Proof: Let us denote by $T : T_x\mathscr{M} \to T_x\mathscr{M}$ one of the endomorphisms considered. Since T maps bases into bases, it must be a vector space isomorphism so that it can be represented by an element of $GL\left(T_x\mathscr{M}\right)$ as the invertible matrix

$$T \equiv \begin{pmatrix} a & \mathbf{b} \\ \mathbf{c}^T & \mathbf{R} \end{pmatrix} \tag{3.2.11}$$

where $a \in \mathbb{R}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$ and $\mathbf{R} \in GL\left(\mathbb{R}^d\right)$. Let $B_x = \{N_x, e_{ix}\}$ be a Galilean basis of $T_x\mathscr{M}$, the basis $T\left(B_x\right) = \{N'_x, e'_{ix}\}$ reads (dropping the index x for notational simplicity):

$$T\begin{pmatrix} N \\ e_i \end{pmatrix} = \begin{pmatrix} a & \mathbf{b} \\ \mathbf{c}^T & \mathbf{R} \end{pmatrix}\begin{pmatrix} N \\ e_i \end{pmatrix} = \begin{pmatrix} aN + \mathbf{b}^i e_i \\ \mathbf{c}^T_j N + \mathbf{R}^i{}_j e_i \end{pmatrix}. \tag{3.2.12}$$

Requiring that $T\left(B_x\right)$ is a Galilean basis (Conditions 1-3 following Definition 3.2.13) imposes that T satisfy:

1. $\psi_x\left(N'_x\right) = 1 \Rightarrow a = 1$
2. $\psi_x\left(e'_{ix}\right) = 0\,, \forall\, i \in \{1, \ldots, d\} \Rightarrow \mathbf{c}^T_j = 0$
3. $\gamma_x\left(e'_{ix}, e'_{jx}\right) = \delta_{ij}\,, \forall\, i, j \in \{1, \ldots, d\} \Rightarrow \mathbf{R} \in O\left(d\right)$.

The set of matrices representing the set of isomorphisms T is thus of the form

$$T = \begin{pmatrix} 1 & \mathbf{b} \\ 0 & \mathbf{R} \end{pmatrix} \tag{3.2.13}$$

with $\mathbf{b} \in \mathbb{R}^d$ and $\mathbf{R} \in O\left(d\right)$. This set of matrices form a subgroup of $GL\left(\mathbb{R}^{1,d}\right)$ isomorphic to the homogeneous Galilei group Gal_0. The homogeneous Galilei group therefore acts regularly on the space of Galilean basis via the group action:

$$\{N, e_i\} \mapsto \left\{N + \mathbf{b}^i e_i, \mathbf{R}^j{}_i e_j\right\}. \tag{3.2.14}$$

□

Proposition 3.2.14 is important since it constitutes the first occurrence of a link between nonrelativistic groups and nonrelativistic physics, which will become a leitmotif of the second part of this work. Together with Definition 3.2.13, it can be generalised in a straightforward way from the tangent space at a point of $\mathscr{M}$ to the tangent bundle of $\mathscr{M}$. A Galilean basis of $T\mathscr{M}$ is thus defined as the ordered set of fields $B = \{N, e_1, \ldots, e_n\}$ with N a field of observers and $\{e_1, \ldots, e_n\}$ a basis of Ker ψ, orthonormal with respect to γ. Two Galilean bases $\{N', e'_i\}$ and $\{N, e_i\}$ are mapped via a local transformation where $\mathbf{R} : \mathscr{M} \to O(d)$ now parameterises a local rotation and $\mathbf{b}^i : \mathscr{M} \to \mathbb{R}^d$ a local Galilean boost. Explicitly, one has:

$$\begin{cases} N' = N + \mathbf{b}^i e_i \\ e'_i = \mathbf{R}^j_i e_j \end{cases} \tag{3.2.15}$$

where the first expression is a Milne boost (*cf.* Proposition 3.2.10).

As such, a Leibnizian structure does not allow generically a *global* definition of absolute time and space since it only provides a set of *local* clocks and rulers. This drawback can be circumvented by restricting the vector space of possible absolute clocks. The suitable restriction comes in a weak and a strong version. Denoting $\mathcal{D} = \{\mathcal{D}_x\}$ the distribution defined as $\mathcal{D}_x \equiv \text{Ker } \psi_x, \forall x \in \mathscr{M}$, the weak version consists in imposing that the distribution $\mathcal{D}$ is involutive. One is then led to define an Aristotelian structure (*cf.* Section 2.1) as:

Definition 3.2.15 (Aristotelian structure)**.** *An Aristotelian structure is a Leibnizian structure whose absolute clock induces an involutive distribution, i.e.* $\psi \wedge d\psi = 0$.

This supplementary condition ensures, by Frobenius Theorem (*cf.* Theorem A.3.7), that the kernel of ψ defines a foliation of $\mathscr{M}$ by a family of hypersurfaces of codimension one called *simultaneity slices*. These are the maximal integral submanifolds of $\mathcal{D}$, so that the tangent space $T_x\mathscr{M}$ at each point x of the simultaneity slice is isomorphic to Ker ψ_x. Locally, the 1-form ψ can be written as $\psi = \Omega\, dt$ where $\Omega \in C^\infty(\mathscr{M})$ is a positive function called *time unit* and the function $t \in C^\infty(\mathscr{M})$ will be referred to as the *absolute time*. The simultaneity slices are the hypersurfaces of fixed absolute time and are thus also called *absolute spaces*. In contradistinction with $\mathscr{M}$, absolute spaces are Riemannian manifolds since they are endowed with the positive-definite metric γ.

Now, let $\mathscr{C}$ be a 1-dimensional immersed submanifold of $\mathscr{M}$ defined by the injective immersion $i : \mathscr{C} \to \mathscr{M}$. The local condition $\psi = \Omega\, dt$ allows to write $d\tau = (\Omega \circ i)\, d(t \circ i)$, with $\tau \in C^\infty(\mathscr{M})$ a proper time on $\mathscr{C}$. Integrating this equality, any observer on an Aristotelian

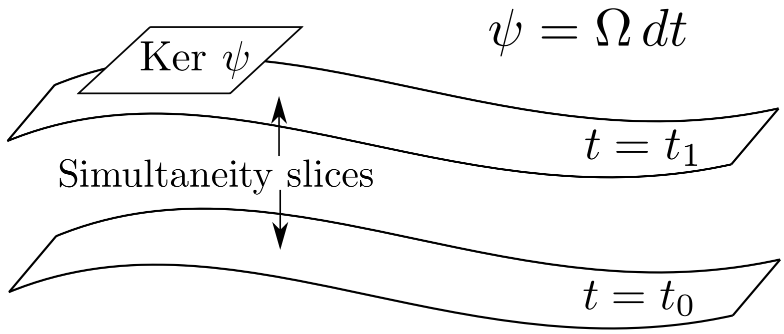


Figure 3.1: Foliation of an Aristotelian structure by simultaneity slices.

structure can thus make use of the time unit Ω in order to compare or "synchronise" its proper time τ with the absolute time t.

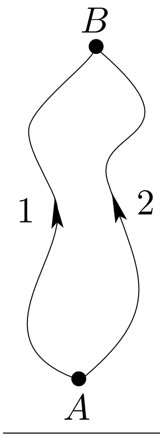
The situation regarding synchronisation is even clearer when considering the more restrictive case in which the absolute clock is a closed 1-form. We thus define an *Augustinian structure*[7] as:

Definition 3.2.16 (Augustinian structure)**.** *An Augustinian structure is a Leibnizian structure whose absolute clock ψ is closed.*

Example 3.2.17 (Augustinian spacetime). The most simple example of a Leibnizian structure is given by a $(d+1)$-dimensional Augustinian spacetime characterised by a closed absolute clock ψ and a flat spatial metric γ:

$$\begin{cases} \psi = dt \\ \gamma = \delta_{ij} dx^i \vee dx^j \end{cases}$$

where $i, j \in \{1, \dots, d\}$ and δ_{ij} the Kronecker delta.



This stronger condition allows locally to write $\psi = dt$, so that any observer of an Augustinian structure is automatically synchronised with the absolute time ($\tau = t \circ i$). Consequently, two observers sharing the same endpoints $A, B \in \mathscr{M}$ will agree when comparing the proper time passed when going from A to B, since the integral

$$\tau_{A\to B} = \int_{\substack{A \\ \mathscr{C}}}^{B} i^*\psi = \int_{\substack{A \\ \mathscr{C}}}^{B} d\left(t \circ i\right) = t\left(i\left(B\right)\right) - t\left(i\left(A\right)\right)$$

does not depend on the path followed.

7. We chose to refer to Augustine of Hippo in order to pay tribute to the role he played regarding the philosophy of time.

3.2.2 Nonrelativistic manifolds

We now switch to the definition of nonrelativistic manifolds, *i.e.* Leibnizian structures endowed with a compatible Koszul connection and discuss some peculiarities arising, in contradistinction with the relativistic case sketched in Section 3.1.

Galilean manifolds

It should first be noted that the compatibility condition must apply to the whole metric-like structure, *i.e.* to the absolute rulers *and* clock. One then defines:

Definition 3.2.18 (Galilean manifold [18])**.** *A Leibnizian structure $\mathscr{L}(\mathscr{M}, \psi, h)$ supplemented with a Koszul connection ∇ compatible with the absolute clock ψ and with the contravariant metric h, i.e. satisfying*

1. $\nabla \psi = 0$
2. $\nabla h = 0$

is called a Galilean manifold and is denoted $\mathscr{G}(\mathscr{M}, \psi, h, \nabla)$.

These two conditions can be more explicitly stated as:

1. $X[\psi(Y)] = \psi(\nabla_X Y)$, $\forall\, X, Y \in \Gamma(T\mathscr{M})$
2. $X[h(Y, Z)] = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$, $\forall\, X, Y, Z \in \Gamma(T\mathscr{M})$.

Similarly to the Leibnizian case, an alternative definition can be formulated as follows:

Definition 3.2.19 (Galilean manifold [92])**.** *A Leibnizian structure $\mathscr{L}(\mathscr{M}, \psi, \gamma)$ supplemented with a Koszul connection ∇ compatible with the absolute clock ψ and with the set of rulers γ, i.e. satisfying*

1. $\nabla \psi = 0$
2. $\nabla \gamma = 0$

is called a Galilean manifold and is denoted $\mathscr{G}(\mathscr{M}, \psi, \gamma, \nabla)$.

Condition *2.* can be reformulated as:

2. $X[\gamma(V, W)] = \gamma(\nabla_X V, W) + \gamma(V, \nabla_X W)$, $\forall\, X \in \Gamma(T\mathscr{M})$ and $\forall\, V, W \in \text{Ker}\ \psi$.

The right side of the previous equation is well defined as $Y \in \text{Ker}\ \psi$ implies $\psi(\nabla_X Y) = 0$ (*cf.* Condition *1.*) which in turn, ensures that $\nabla_X Y \in \text{Ker}\ \psi$, $\forall\, X \in \Gamma(T\mathscr{M})$.

In components, these two sets of equivalent conditions read:

$$\begin{cases}\nabla_\mu \psi_\nu = 0\\ \nabla_\mu \gamma_{\alpha\beta} = 0\end{cases} \Longleftrightarrow \begin{cases}\nabla_\mu \psi_\nu = 0\\ \nabla_\mu h^{\alpha\beta} = 0\end{cases}. \tag{3.2.16}$$

A first peculiarity of a Galilean manifold, in contradistinction with the relativistic case, is the fact that not all the torsion 2-forms are compatible with a given Leibnizian structure, as the following Proposition shows:

Proposition 3.2.20 (*cf.* [18, 92])**.** *Let $\mathscr{G}(\mathscr{M},\psi,\gamma,\nabla)$ be a Galilean manifold and denote T the torsion of ∇. Then we have:*

$$\psi\left(T\left(X,Y\right)\right) = d\psi\left(X,Y\right)$$

for all $X,Y\in\Gamma\left(T\mathscr{M}\right)$.

In components, this relation reads $\psi_\lambda T^\lambda_{\mu\nu} = \partial_{[\mu}\psi_{\nu]}$, where the torsion T (*cf.* Definition A.9.2) decomposes in holonomic coordinates as $T \equiv T^\lambda_{\mu\nu}\, dx^\mu \wedge dx^\nu \otimes \partial_\lambda$.

Proof: Starting with the definition of the torsion (*cf.* eq.(A.9.13)), and acting with ψ on both sides:

$$\begin{aligned}\psi\left(T\left(X,Y\right)\right) &= \psi\left(\nabla_X Y\right) - \psi\left(\nabla_Y X\right) - \psi\left([X,Y]\right)\\ &= X\left[\psi\left(Y\right)\right] - Y\left[\psi\left(Y\right)\right] - \psi\left([X,Y]\right)\\ &= d\psi\left(X,Y\right)\end{aligned}$$

where in the first step, condition *1.* of Definition 3.2.18 has been used. □

In particular, one sees that only Augustinian structures ($d\psi = 0$) admit a torsionfree Koszul connection. This is clearly a distinctive feature of nonrelativistic structures as there exists no such restriction in the relativistic case. Furthermore, while in the relativistic case, Corollary 3.1.5 ensures that a torsionfree Lorentzian manifold is uniquely determined by the metric structure, in the nonrelativistic case, however, as the degeneracy of the metric prevents Corollary 3.1.5 to hold, the gift of an Augustinian structure does not uniquely fix a compatible torsionfree Koszul connection.

Example 3.2.21 (Galilean and Newton-Hooke spacetimes)*.* The Augustinian spacetime of Example 3.2.17 can be supplemented with the flat connection in order to yield the standard *flat Galilean spacetime*:

$$\begin{cases} \psi = dt \\ \gamma = \delta_{ij} dx^i \vee dx^j \\ \Gamma^\lambda_{\mu\nu} = 0. \end{cases}$$

Alternatively, one can endow the Augustinian spacetime with the (equally compatible) connection $\bar{\Gamma}$:

$$\begin{cases} \psi = dt \\ \gamma = \delta_{ij} dx^i \vee dx^j \\ \bar{\Gamma}^i_{00} = -\frac{k}{\tau^2} x^i \end{cases}$$

where the other components of $\bar{\Gamma}$ vanish. This manifold is referred to as the *Newton-Hooke spacetime*. The constant k can take the values $+1$ (expanding spacetime) or -1 (oscillating spacetime).

As illustrated in the previous example, several Koszul connections are admissible, once given a metric structure. In fact, to a given Augustinian structure corresponds a whole class of compatible torsionfree Koszul connections, as expressed in the following Theorem:

Theorem 3.2.22 ([17, 18])**.** *Given a field of observers $N \in FO(\mathscr{M})$, the set of torsionfree Galilean manifolds compatible with a given Augustinian structure $\mathscr{S}(\mathscr{M}, \psi, \gamma)$ is in bijective correspondence with the set $\Omega^2(\mathscr{M})$ of 2-forms $\overset{N}{F}$ on $\mathscr{M}$.*

Given a Galilean connection ∇ and a field of observers $N \in FO(\mathscr{M})$, the corresponding 2-form $\overset{N}{F} \in \Omega^2(\mathscr{M})$ is defined as

$$\overset{N}{F}(X, Y) \equiv \gamma\left(\nabla_X N, P^N(Y)\right) - \gamma\left(\nabla_Y N, P^N(X)\right) \tag{3.2.17}$$

where $X, Y \in \Gamma(T\mathscr{M})$ are vector fields on $\mathscr{M}$, $\gamma \in$ and P^N designates the spacelike projector (*cf.* Definition 3.2.9). In holonomic coordinates, eq.(3.2.17) reads

$$\overset{N}{F}_{\alpha\beta} \equiv -2\overset{N}{\gamma}_{\lambda[\alpha} \nabla_{\beta]} N^\lambda.$$

Under a Milne boost $N \to N + h(\chi)$, $\overset{N}{F}$ transforms as $\overset{N}{F} \to \overset{N}{F} + d\Phi$ where the 1-form $\Phi \in \Omega^1(\mathscr{M})$ is defined as

$$\Phi \equiv \chi - \left(\chi(N) + \frac{1}{2} h(\chi, \chi)\right) \psi. \tag{3.2.18}$$

We then define an equivalence class $\left[N, \overset{N}{F}\right]$ as follows: two couples $\left(N', \overset{N'}{F}\right)$ and $\left(N, \overset{N}{F}\right)$ are said to be equivalent if there exists a 1-form $\chi \in \Omega^1\left(\mathscr{M}\right)$ such that

$$\begin{cases} N' = N + h\left(\chi\right) \\ \overset{N'}{F} = \overset{N}{F} + d\Phi \end{cases} \tag{3.2.19}$$

where the 1-form Φ is expressed in terms of the 1-form χ as $\Phi_\mu = \chi_\mu - \big(\chi\left(N\right) + \frac{1}{2}h\left(\chi,\chi\right)\big)\,\psi_\mu$.

Therefore, torsionfree Galilean connections preserving a given Augustinian structure are seen to be in one-to-one correspondence with equivalence classes of $\left[N, \overset{N}{F}\right]$.

In holonomic coordinates, one writes $\nabla_\mu Y^\lambda = \partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu}Y^\nu$, where the components $\Gamma^\lambda_{\mu\nu}$ read [23]:

$$\Gamma^\lambda_{\mu\nu} = N^\lambda \partial_{(\mu}\psi_{\nu)} + \frac{1}{2}h^{\lambda\rho}\left[\partial_\mu \overset{N}{\gamma}_{\rho\nu} + \partial_\nu \overset{N}{\gamma}_{\rho\mu} - \partial_\rho \overset{N}{\gamma}_{\mu\nu}\right] + h^{\lambda\rho}\psi_{(\mu}\overset{N}{F}_{\nu)\rho}. \tag{3.2.20}$$

Indeed, one can check that this expression is invariant under a Milne boost, so that it is independent of the choice of representative in the equivalence class $\left[N, \overset{N}{F}\right]$.

Expressing the 2-form $\overset{N}{F}$ on the Galilean basis $\left(N, e_i\right)$ (with $\left(\psi, \theta^i\right)$ the associated dual basis) leads to the following decomposition:

$$\overset{N}{F} = 2\overset{N}{F}\left(N, e_i\right)\psi \wedge \theta^i + \overset{N}{F}\left(e_i, e_j\right)\theta^i \wedge \theta^j. \tag{3.2.21}$$

The first term defines a spacelike vector field $\overset{N}{\mathbf{G}} \in \text{Ker } \psi$ as $\overset{N}{\mathbf{G}} = \overset{N}{F}\left(N, e_i\right)e^i$ (where $e^i \equiv e_j\delta^{ij}$) designated as the gravitational field. The second term corresponds to the action of $\overset{N}{F}$ on spacelike vector fields and will be referred to as the Coriolis 2-form $\overset{N}{\boldsymbol{\omega}} \in \Omega^2\left(\text{Ker } \psi\right)$ defined as $\overset{N}{\boldsymbol{\omega}}\left(V, W\right) = \overset{N}{F}\left(V, W\right)$, with $V, W \in \text{Ker } \psi$. Using eq.(3.2.17), these two definitions can be recast in a more geometric way which justifies further the terminology used:

Definition 3.2.23 (Gravitational field and Coriolis 2-form [92]). *Let $\mathscr{G}\left(\mathscr{M}, \psi, \gamma, \nabla\right)$ be a Galilean manifold and $N \in FO\left(\mathscr{M}\right)$ a field of observers. The gravitational field induced by ∇ on N is the spacelike vector field $\overset{N}{\boldsymbol{G}} \in \text{Ker } \psi$:*

$$\overset{N}{\boldsymbol{G}} \equiv \nabla_N N. \tag{3.2.22}$$

The Coriolis 2-form induced by ∇ *on* N *is the 2-form* $\overset{N}{\boldsymbol{\omega}} \in \Omega^2(\text{Ker }\psi)$*, acting on* $V, W \in$ Ker ψ *as*[8]*:*

$$\overset{N}{\boldsymbol{\omega}}(V, W) \equiv \gamma(\nabla_V N, W) - \gamma(V, \nabla_W N). \tag{3.2.23}$$

The compatibility condition of the Galilean connection ∇ with the absolute clock ψ (*cf.* Condition 1 of Definition 3.2.19) ensures that $\psi(\nabla_X N) = X[\psi(N)] = 0$, $\forall X \in \Gamma(T\mathscr{M})$. This expression ensures $\psi(\nabla_N N) = 0$, which in turn guarantees that $\overset{N}{\mathbf{G}}$ is spacelike.

According to the decomposition (3.2.21), the gravitational field $\overset{N}{\mathbf{G}}$ and the Coriolis 2-form $\overset{N}{\boldsymbol{\omega}}$ associated to the field of observers N encode all the information contained in the 2-form $\overset{N}{F}$. Hence, given a field of observers N, torsionfree Galilean manifolds compatible with a given Augustinian structure can equivalently be put in bijective correspondence with couples $\left(\overset{N}{\mathbf{G}}, \overset{N}{\boldsymbol{\omega}}\right)$, $\overset{N}{\mathbf{G}} \in \text{Ker }\psi$, $\overset{N}{\boldsymbol{\omega}} \in \Omega^2(\text{Ker }\psi)$ (*cf.* Corollary 5.28 in [92]).

Curvature of a torsionfree Galilean manifold

This paragraph is devoted to the study of the curvature tensor for a Galilean manifold. We will establish some useful identities, focusing on the torsionfree case and discuss classic constraints encountered in the literature.

Let us restate expression (A.9.14) for the Koszul curvature:

$$R(X, Y; f) = \nabla_X \nabla_Y f - \nabla_Y \nabla_X f - \nabla_{[X,Y]} f$$

with $X, Y \in \Gamma(T\mathscr{M})$ and $f \in \Gamma(E)$. In holonomic coordinates, the components of the Koszul curvature read: $dx^\lambda[R(\partial_\mu, \partial_\nu; \partial_\rho)] \equiv R^\lambda{}_{\rho\mu\nu}$.

Compatibility conditions (3.2.16) for the Galilei connection ∇ impose the following constraints on the Koszul curvature:

$$\begin{cases} \nabla_\mu \psi_\nu = 0 \Rightarrow \psi_\lambda R^\lambda{}_{\rho\mu\nu} = 0 \\ \nabla_\mu h^{\alpha\beta} = 0 \Rightarrow h^{\rho\beta} R^\alpha{}_{\rho\mu\nu} + h^{\alpha\rho} R^\beta{}_{\rho\mu\nu} = 0 \end{cases}$$

Notation 3.2.24. In the following we will use a Galilean basis $B \equiv \{N, e_i\}$ together with its dual $B^* \equiv \{\psi, \theta^i\}$. Now, let T^μ_ν be the holonomic components of a tensor $T \in$

8. Note that our normalisation for the Coriolis 2-form differs by a factor $\frac{1}{2}$ from the one used in [92].

$\Gamma\left(T\mathscr{M}\times T^*\mathscr{M}\right)$. The following notation will prove to be handy:

$$\begin{cases} T^0_\nu \equiv \psi_\mu T^\mu_\nu & T^i_\nu \equiv \theta^i_\mu T^\mu_\nu \\ T^\mu_0 \equiv N^\nu T^\mu_\nu & T^\mu_i \equiv e^\nu_i T^\mu_\nu . \end{cases}$$

The precedently stated constraints on the Koszul curvature can thus be reexpressed as:

$$R^0{}_{\rho\mu\nu} = R^{(ij)}{}_{\mu\nu} = 0. \tag{3.2.24}$$

Taking these constraints into account, the components of the curvature 2-form $R^\lambda{}_\rho \in \Omega^2\left(\mathscr{M}\right)$ can be expanded as:

$$R^\lambda{}_\rho = R^i{}_j\theta^j_\rho e^\lambda_i + R^i{}_0\psi_\rho e^\lambda_i. \tag{3.2.25}$$

Proposition 3.2.25 (Symmetries of the Galilean curvature)**.** *The Galilean curvature tensor satisfies the following identities:*

$$\begin{cases} R^i{}_{\rho(\mu\nu)} = 0 \\ R^i{}_{[\rho\mu\nu]} = 0 \\ R^i{}_k{}^j{}_l = R^j{}_l{}^i{}_k . \end{cases} \tag{3.2.26}$$

Proof: These identities follow respectively from eq.(A.9.15), (A.9.16) and (A.9.18) of Proposition A.9.5 applied to the spatial metric $\gamma \in \Gamma\left(\vee^2 \operatorname{Ker}\psi\right)$. $\square$

The second identity of the previous Proposition, known as the first Bianchi identity, decomposes further into the following set:

$$\begin{cases} R^l{}_{[ij]0} + \frac{1}{2}R^l{}_{0ij} = 0 \\ R^l{}_{[ijk]} = 0. \end{cases} \tag{3.2.27}$$

Proposition 3.2.26. *The first Bianchi identity for the Galilei curvature leads to the following set of equations:*

$$\begin{cases} R^l{}_{[ij]0} + \frac{1}{2}R^l{}_{0ij} = 0 \\ R^l{}_{[ijk]} = 0. \end{cases} \tag{3.2.28}$$

Proof: The first Bianchi identity (*cf.* eq. (A.9.16)) admits a formulation in terms of differential forms as $R^{\lambda}{}_{\rho} \wedge dx^{\rho} = 0$ which can be expressed in the Galilean basis as

$$R^{\lambda}{}_{0} \wedge \psi + R^{\lambda}{}_{i} \wedge \theta^{i} = 0. \tag{3.2.29}$$

Expressing the curvature 2-form $R^{\lambda}{}_{\rho}$ in the Galilean 2-forms basis as

$$R^{\lambda}{}_{\rho} = \frac{1}{2}\left[R^{\lambda}{}_{\rho 0 i}\psi \wedge \theta^{i} + \frac{1}{2}R^{\lambda}{}_{\rho i j}\theta^{i} \wedge \theta^{j}\right] \tag{3.2.30}$$

and plugging back into eq.(3.2.29) leads to:

$$\left[R^{\lambda}{}_{j0i} + \frac{1}{2}R^{\lambda}{}_{0ij}\right]\psi \wedge \theta^{i} \wedge \theta^{j} + \frac{1}{2}R^{\lambda}{}_{ijk}\theta^{i} \wedge \theta^{j} \wedge \theta^{k} = 0. \tag{3.2.31}$$

Taking into account the compatibility conditions (3.2.24) as well as the antisymmetry $R^{\lambda}{}_{\rho\mu\nu} = R^{\lambda}{}_{\rho[\mu\nu]}$ (*cf.* eq.(A.9.15)) gives the expected result. $\square$

Corollary 3.2.27 (*cf.* [72])**.** *The Galilei curvature satisfies:*

$$R^{(i}{}_{[\mu}{}^{j)}{}_{\nu]} = 0. \tag{3.2.32}$$

One way to partially reduce the ambiguity in the definition of the torsionfree Galilean connection is to impose supplementary conditions. The following condition [18, 53] has been proved very useful:

Definition 3.2.28 (Duval-Künzle condition)**.** *Let* $\mathscr{G}(\mathscr{M}, \psi, \gamma, \nabla)$ *be a Galilean manifold and denote* R *the curvature of the Galilean connection* ∇*. The Duval-Künzle condition then reads:*

$$\alpha\Big(R(X, h(\beta); Y)\Big) = \beta\Big(R(Y, h(\alpha); X)\Big) \tag{3.2.33}$$

$\forall X, Y \in \Gamma(T\mathscr{M})$ *and* $\alpha, \beta \in \Omega^{1}(\mathscr{M})$*.*

This condition on the curvature operator R (*cf.* Definition A.9.3) writes more transparently in components as:

$$R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} = R^{\nu}{}_{\beta}{}^{\mu}{}_{\alpha}$$

with $R^{\mu\ \ \nu}_{\ \alpha\ \ \beta} \equiv h^{\nu\rho}\, R^{\mu}_{\ \alpha\rho\beta}$. Decomposing on a Galilean basis leads to the set of equations:

$$\begin{cases} R^{i\ \ j}_{\ k\ \ l} = R^{j\ \ i}_{\ l\ \ k} \\ R^{[i\ \ j]}_{\ \ 0\ \ 0} = 0 \\ R^{j\ \ i}_{\ 0\ \ k} = R^{i\ \ j}_{\ k\ \ 0} \end{cases}$$

The first equation is already implied by the first Bianchi identity (*cf.* eq.(3.2.27)). However the two remaining are non-trivial constraints that reduce the number of independent components from $\frac{1}{12}d^2\,(d+1)\,(d+5)$ to $\frac{1}{12}\,(d+1)^2\left[(d+1)^2-1\right]$, *i.e.* to the same number as in a $(d+1)$-dimensional pseudo-Riemannian manifold (*cf.* *e.g.* [18]).

Proposition 3.2.29. *The Duval-Künzle condition can be alternatively written as*

$$R^{i}_{\ 0}\wedge\theta_i = 0. \tag{3.2.34}$$

Proof: This alternative formulation imposes the following constraints:

$$\begin{cases} R^{[i\ \ j]}_{\ \ 0\ \ 0} = 0 \\ R^{[i\ \ k\ l]}_{\ \ 0} = 0. \end{cases}$$

The first equality matches the second one from Definition 3.2.28 so what remains to be proved is the following equivalence:

$$R^{j\ \ i}_{\ 0\ \ k} = R^{i\ \ j}_{\ k\ \ 0} \Leftrightarrow R^{[i\ \ j\ k]}_{\ \ 0} = 0. \tag{3.2.35}$$

We start by totally antisymmetrising the first of the identities of the Bianchi set (3.2.28):

$$R^{[i\,j\,k]}_{\ \ \ \ \ \ 0} + \frac{1}{2}R^{[i\ \ j\ k]}_{\ \ 0} = 0. \tag{3.2.36}$$

Expanding the first term leads to:

$$\frac{1}{3}\left(2R^{j\,[k\,i]}_{\ \ \ \ \ \ 0} - R^{i\,k\,j}_{\ \ \ \ \ \ 0}\right) + \frac{1}{2}R^{[i\ \ j\ k]}_{\ \ 0} = 0. \tag{3.2.37}$$

Reusing the first Bianchi identity allows to transform the first term on the left-hand side:

$$\frac{1}{3}\left(R^{j\ \ i\ k}_{\ 0} - R^{i\,k\,j}_{\ \ \ \ \ \ 0}\right) + \frac{1}{2}R^{[i\ \ j\ k]}_{\ \ 0} = 0. \tag{3.2.38}$$

so that $R^{j\ \ i}_{\ 0\ \ k} = R^{i\ \ j}_{\ k\ \ 0} \Leftrightarrow R^{[i\ \ j\ k]}_{\ \ 0} = 0.$ □

Along the Duval-Künzle condition, another constraint on the curvature, dubbed the Trautman condition [9] is frequently encountered in the literature:

Definition 3.2.30 (Trautman condition, *cf.* *e.g.* [1]). *Let J be the Jacobi curvature operator defined as*

$$J\left(X,Y;Z\right)\equiv\frac{1}{2}\left(R\left(Z,X;Y\right)+R\left(Z,Y;X\right)\right) \tag{3.2.39}$$

where X, Y and Z are three vector fields. The Trautman condition thus states that the Jacobi operator must be self-dual when acting on spacelike vectors, i.e.

$$\gamma\left(J\left(X,Y;v\right),w\right)=\gamma\left(J\left(X,Y;w\right),v\right) \tag{3.2.40}$$

with $X,Y\in\Gamma\left(T\mathscr{M}\right)$ and $v,w\in\operatorname{Ker}\psi$.

In components, the Jacobi operator reads $J^{\lambda}{}_{\rho\mu\nu}=R^{\lambda}{}_{(\mu|\rho|\nu)}$ while the Trautman condition imposes:

$$R^{[i\ \ j]}{}_{(\mu\ \ \nu)}=0. \tag{3.2.41}$$

Proposition 3.2.31 (*cf.* [72]). *The Duval-Künzle and Trautman conditions are equivalent for a torsionfree Galilean manifold.*

Proof: One starts by establishing the following Lemma:

Lemma 3.2.32. *The curvature tensor of a torsionfree Galilean manifold satisfies the relation:*

$$R^{(i\ \ j)}{}_{[\mu\ \ \nu]}=0$$

Proof: The relation is equivalent to the set:

$$\begin{cases}R^{(i\ \ j)}{}_{[k\ \ l]}=0\\ R^{(i\ \ j)}{}_{[0\ \ k]}=0.\end{cases}$$

The first relation follows straightforwardly from the all-spacelike first Bianchi identity and compatibility relations (3.2.24). The second identity is obtained

9. Although the denominations Duval-Künzle and Trautman conditions seem customary in the literature, it is amusing to note that in the respective works commonly cited when these conditions are discussed, Trautman writes what is usually referred to as the Duval-Künzle condition (*cf.* eq.(IV) of [15]) while Künzle writes the Trautman condition (*cf.* eq.(4.14) of [18]).

by taking the symmetric part in $(l \leftrightarrow i)$ of the temporal/spacelike Bianchi identity:

$$R_{(li)j0} - R_{(l|j|i)0} + R_{(l|0|i)j} = 0. \tag{3.2.42}$$

The first term vanishes, leaving $R^{(l\ \ i)}{}_{[0\ \ j]} = 0$. □

Now, decomposing the Duval-Künzle operator as:

$$\begin{aligned}
\underbrace{R^{i}{}_{\mu}{}^{j}{}_{\nu} - R^{j}{}_{\nu}{}^{i}{}_{\mu}}_{\text{(DK)}} &= \frac{1}{2}\left(R^{(i}{}_{\mu}{}^{j)}{}_{\nu} + R^{[i}{}_{\mu}{}^{j]}{}_{\nu} - R^{(j}{}_{\nu}{}^{i)}{}_{\mu} - R^{[j}{}_{\nu}{}^{i]}{}_{\mu}\right) \\
&= \frac{1}{2}\left(R^{(i}{}_{\mu}{}^{j)}{}_{\nu} + R^{[i}{}_{\mu}{}^{j]}{}_{\nu} - R^{(i}{}_{\nu}{}^{j)}{}_{\mu} + R^{[i}{}_{\nu}{}^{j]}{}_{\mu}\right) \\
&= \underbrace{R^{(i\ \ j)}{}_{[\mu\ \ \nu]}}_{\text{(L)}} + \underbrace{R^{[i\ \ j]}{}_{(\mu\ \ \nu)}}_{\text{(T)}}
\end{aligned} \tag{3.2.43}$$

one recognises the operator (L) obtained in Lemma 3.2.32 as well as the Trautman operator (T). Provided Lemma 3.2.32, the Duval-Künzle and Trautman conditions are therefore equivalent. □

Newtonian manifolds

We now focus our attention to the study of torsionfree Galilean manifolds satisfying the Duval-Künzle condition (*cf.* Definition 3.2.28).

Definition 3.2.33 (Newtonian manifold)**.** *A Newtonian manifold $\mathscr{N}(\mathscr{M}, \psi, \gamma, \nabla)$ is a torsionfree Galilean manifold whose compatible Koszul connection satisfies the Duval-Künzle condition.*

Theorem 3.2.34 ([18, 53])**.** *Given a field of observers $N \in FO(\mathscr{M})$, the set of Newtonian manifolds compatible with a given Augustinian structure $\mathscr{S}(\mathscr{M}, \psi, \gamma)$ is in bijective correspondence with the set of closed 2-forms $\overset{N}{F}$.*

In the light of the previous Theorem, the Duval-Künzle condition can be reinterpreted as a geometric characterisation for the closedness of the 2-forms $\overset{N}{F}$ belonging to the equivalence class $\left[N, \overset{N}{F}\right]$ characteristic of the Galilean manifold. Applying Poincaré Lemma, one can locally write a given $\overset{N}{F}$ as an exact form so that there exists a class of 1-forms $\overset{N}{A} \in \Omega^1(\mathscr{M})$ satisfying $\overset{N}{F} = d\overset{N}{A}$. Two equivalent 1-forms $\overset{N}{A}{}'$ and $\overset{N}{A}$ differ by an exact differential: $\overset{N}{A}{}' = \overset{N}{A} + df$, with $f \in C^\infty(\mathscr{M})$. The transformation $\overset{N}{A} \to \overset{N}{A} + df$ will be referred to as a Maxwell-gauge transformation.

On the other hand, the transformation law of a 1-form $\overset{N}{A}$ under a Milne boost $N \to N + h(\chi)$ follows directly from the one for $\overset{N}{F}$ and is given by $\overset{N}{A} \to \overset{N}{A} + \Phi$ with Φ as in (3.2.18). Again, one can define an equivalence class $\left[N, \overset{N}{A}\right]$ as follows: two couples $\left(N', \overset{N'}{A}\right)$ and $\left(N, \overset{N}{A}\right)$ are said to be equivalent if there exists a 1-form $\chi \in \Omega^1(\mathscr{M})$ and a function $f \in C^\infty(\mathscr{M})$ such that

$$\begin{cases} N' = N + h(\chi) \\ \overset{N'}{A} = \overset{N}{A} + \Phi + df. \end{cases} \tag{3.2.44}$$

A Newtonian manifold is thus (locally) determined by an Augustinian structure supplemented with an equivalence class $\left[N, \overset{N}{A}\right]$. The different structures necessary to uniquely determine a given manifold are summarised in the following table, both in the relativistic and nonrelativistic cases:

Metric structure	Supplementary structure	Manifold
Lorentzian $(\mathscr{M}, g)$	$\times$	Lorentzian
Augustinian $(\mathscr{M}, \psi, \gamma)$	Gravitational field strength $\left[N, \overset{N}{F}\right]$	Galilean
	Gravitational potential $\left[N, \overset{N}{A}\right]$	Newtonian

Lagrangian structures

In this Section, we revisit the equivalence problem for Newtonian manifolds (*i.e.* the search for extensions of a given Augustinian structure determining uniquely a Newtonian connection) by displaying an alternative formulation [23], based on Coriolis-free fields of observers (*cf.* Definition 3.2.23). We start by proving the following Proposition:

Proposition 3.2.35. *Let $\mathscr{N}(\mathscr{M}, \psi, \gamma, \nabla)$ be a Newtonian manifold associated to the equivalence class $\left[N, \overset{N}{A}\right]$. The field of observers $Z \in FO(\mathscr{M})$ is Coriolis-free if and only if there* *exists a function* $f \in C^{\infty}(\mathscr{M})$ *such that*

$$Z = N - h\left(\overset{N}{A}\right) + h\left(df\right) \tag{3.2.45}$$

where $\left(N, \overset{N}{A}\right)$ *is any representative of the equivalence class.*

Proof: Let us first check that the previous definition for Z is well-defined under a change of representative (*cf.* eq.(3.2.44)). This is easily seen as:

$$\begin{aligned} Z' &= N' - h\left(\overset{N'}{A}\right) + h\left(df'\right) \\ &= N + h(\chi) - h\left(\overset{N}{A} + \Phi + df\right) + h\left(df'\right) \\ &= N - h\left(\overset{N}{A}\right) + h\Big(d\left(f' - f\right)\Big) \end{aligned}$$

so the previous definition is consistent. Now, let us compute the Coriolis 2-form of a field of observers $Z = N + h(\chi)$, with $\chi \in \Omega^1(\mathscr{M})$:

$$\begin{aligned} \overset{Z}{\boldsymbol{\omega}}(V, W) &= \gamma\left(\nabla_V Z, W\right) - \gamma\left(V, \nabla_W Z\right) \\ &= \overset{N}{\boldsymbol{\omega}}(V, W) + \gamma\left(\nabla_V h(\chi), W\right) - \gamma\left(\nabla_W h(\chi), V\right) \end{aligned}$$

with $V, W \in \text{Ker } \psi$. Note that the second and third terms make sense, since $\psi\big(\nabla_V h(\chi)\big) = V\left[\psi\big(h(\chi)\big)\right] = 0$. Using $\nabla\gamma = 0$ allows to reformulate the first term in brackets as $\gamma\left(\nabla_V h(\chi), W\right) = V\left[\gamma\left(h(\chi), W\right)\right] - \gamma\left(h(\chi), \nabla_V W\right)$. Proceeding similarly with the second term in brackets leads to:

$$\begin{aligned} \overset{Z}{\boldsymbol{\omega}}(V, W) &= \overset{N}{\boldsymbol{\omega}}(V, W) + \Big(V\left[\gamma\left(h(\chi), W\right)\right] - \gamma\left(h(\chi), \nabla_V W\right) - (V \leftrightarrow W)\Big) \\ &= \overset{N}{\boldsymbol{\omega}}(V, W) + V\left[\chi(W)\right] - W\left[\chi(V)\right] - \chi\left(\nabla_V W - \nabla_W V\right) \\ &= \overset{N}{\boldsymbol{\omega}}(V, W) + V\left[\chi(W)\right] - W\left[\chi(V)\right] - \chi\left([V, W]\right) \\ &= \overset{N}{\boldsymbol{\omega}}(V, W) + d\chi(V, W) \end{aligned}$$

where one used respectively: in the first step, the equality $\gamma\left(h(\alpha), X\right) = \alpha(X)$, with $\alpha \in \Omega^1(\mathscr{M})$ and $X \in \text{Ker } \psi$; in the second step, the fact that the Newtonian connection is torsionfree; in the third step, the definition of the exterior derivative of a 1-form.

Imposing that $\overset{Z}{\boldsymbol{\omega}}$ vanishes and using the local expression of $\overset{N}{\boldsymbol{\omega}}$ as $\overset{N}{\boldsymbol{\omega}}(V,W) = d\overset{N}{A}(V,W)$ leads to the condition:

$$d\left(\overset{N}{A}+\chi\right)(V,W)=0 \quad , \quad \forall\, V,W\in \text{Ker }\psi. \tag{3.2.46}$$

Using the involutivity of the distribution induced by Ker ψ allows to locally rewrite condition 3.2.46 as

$$\exists f\in C^\infty(\mathscr{M}) \quad / \quad \chi(V)=-\overset{N}{A}(V)+df(V) \quad , \quad \forall V\in \text{Ker }\psi.$$

Therefore, there exists a function f on $\mathscr{M}$ such that $Z=N-h\left(\overset{N}{A}\right)+h(df)$. □

In the following, we let $\mathscr{N}(\mathscr{M},\psi,\gamma,\nabla)$ be a Newtonian manifold associated to the equivalence class $\left[N,\overset{N}{A}\right]$. As mentioned in the proof, the quantity $Z=N-h\left(\overset{N}{A}\right)$ is invariant under a Milne boost, so that the 1-form $\overset{N}{A}$ can be understood as a compensator field which can be used in order to construct Milne-invariant objects (*cf.* Table 3.1 below). Secondly, Proposition 3.2.35 ensures that any Newtonian manifold admits Coriolis-free fields of observers and furthermore provides an explicit way to construct them: namely, one can go from any field of observers N to a Coriolis-free field of observers Z via a Milne boost parameterised by the 1-form $-\overset{N}{A}$. Under such a Milne boost, the gravitational potential $\overset{N}{A}$ gets mapped as $\overset{N}{A}\to\overset{Z}{A}\equiv\frac{1}{2}\phi\psi$, where the function $\phi\in C^\infty(\mathscr{M})$ is defined as $\phi\equiv 2\overset{N}{A}(N)-h\left(\overset{N}{A},\overset{N}{A}\right)$. It can be checked that ϕ is also a Milne-invariant object and will be referred to as the *scalar gravitational potential* of Z. This denomination is justified by the form taken by the gravitational potential of a Coriolis-free field of observers $\overset{Z}{\mathbf{G}}\equiv\nabla_Z Z=-\frac{1}{2}h(d\phi)$. The couple $\left(Z,\frac{1}{2}\phi\psi\right)$ is thus a distinguished representative of the equivalence class $\left[N,\overset{N}{A}\right]$ characterising the Newtonian manifold $\mathscr{N}$. Since the whole equivalence class $\left[N,\overset{N}{A}\right]$ can be reconstructed from one of its representatives using relations (3.2.44), one can characterise the Newtonian manifold $\mathscr{N}$ by the couple (Z,ϕ) (supplementing the Augustinian structure $\mathscr{S}(\mathscr{M},\psi,\gamma)$).

In order to make a converse statement, one needs first to acknowledge the fact that, as a consequence of Proposition 3.2.35, a given Newtonian manifold do not define a unique Coriolis-free field of observers but rather a class thereof. Indeed, two Coriolis-free fields of observers Z and $Z'\in FO(\mathscr{M})$ have been seen to be related via a function $f\in C^\infty(\mathscr{M})$ as $Z'=Z-h(df)$. This is a direct consequence of the previously mentioned fact that at a given field of observers N corresponds a class of 1-forms $\left[\overset{N}{A}\right]$ differing by $\overset{N'}{A} = \overset{N}{A} + df$, for some function f on $\mathscr{M}$. Consequently, the respective scalar gravitational potentials ϕ and $\phi' \in C^\infty(\mathscr{M})$ can be checked to be related according to $\phi' = \phi + 2\,df\,(Z) - h\,(df, df)$. Newtonian manifolds thus can be put in correspondence with equivalence classes $[Z, \phi]$, where two couples (Z', ϕ') and (Z, ϕ) are said to be equivalent if there exists a function $f \in C^\infty(\mathscr{M})$ such that

$$\begin{cases} Z' = Z - h\,(df) \\ \phi' = \phi + 2\,df\,(Z) - h\,(df, df) \end{cases} \tag{3.2.47}$$

We sum up this discussion in the following Proposition:

Proposition 3.2.36 (*cf.* [23])**.** *Let $\mathscr{S}(\mathscr{M}, \psi, \gamma)$ be an Augustinian structure. There is a bijective correspondence between Newtonian manifolds $\mathscr{N}(\mathscr{M}, \psi, \gamma, \nabla)$ and equivalence classes $[Z, \phi]$.*

An obvious benefit of the present formulation is that it allows to gather up the supplementary information needed to define Newtonian manifolds in a Milne-invariant equivalence class. Indeed, two representatives of this equivalence class only differ by a Maxwell gauge-transformation, so that the present characterisation of Newtonian manifolds is explicitly Milne invariant. Another interesting feature of this formulation is embodied in the following Proposition:

Proposition 3.2.37. *Let $\mathscr{S}(\mathscr{M}, \psi, \gamma)$ be an Augustinian structure. Let $Z \in FO(\mathscr{M})$ designate a field of observers and $\phi \in C^\infty(\mathscr{M})$ a function on $\mathscr{M}$. There is a bijective correspondence between pairs (Z, ϕ) and covariant metrics $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ satisfying the condition $g\,(X, Y) = \gamma\,(X, Y)$, $\forall\, X, Y \in \Gamma\,(\mathrm{Ker}\ \psi)$.*

Proof: We start by proving the implication $(Z, \phi) \Rightarrow g$:

Lemma 3.2.38. *Let $\mathscr{S}(\mathscr{M}, \psi, \gamma)$ be an Augustinian structure, $Z \in FO(\mathscr{M})$ a field of observers and $\phi \in C^\infty(\mathscr{M})$ a function on $\mathscr{M}$. The metric $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ defined as:*

$$g \equiv \overset{Z}{\gamma} + \phi\,\psi \vee \psi$$

with $\overset{Z}{\gamma}$ the metric transverse to Z, is the only metric satisfying

$$\begin{cases} g\,(Z) = \phi\psi \\ g\,(X, Y) = \gamma\,(X, Y),\ \forall\, X, Y \in \Gamma\,(\mathrm{Ker}\ \psi)\,. \end{cases} \tag{3.2.48}$$

Proof: Let $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ be an arbitrary covariant metric on $\mathscr{M}$. The decomposition of g on the Galilean basis $\{Z, e_i\}$ (with dual basis $\left\{\psi, \theta^i\right\}$) reads:

$$g = g\left(Z, Z\right)\psi \vee \psi + 2g\left(Z, e_i\right)\psi \vee \theta^i + g\left(e_i, e_j\right)\theta^i \vee \theta^j.$$

Requiring that g satisfies the conditions 3.2.48 reduces its expression to:

$$g = \phi\psi \vee \psi + \gamma\left(e_i, e_j\right)\theta^i \vee \theta^j$$

where the second term is nothing but $\overset{Z}{\gamma}$. $\square$

A statement converse to Lemma 3.2.38 can be formulated as follows:

Lemma 3.2.39. *Let $\mathscr{S}\left(\mathscr{M}, \psi, \gamma\right)$ be an Augustinian structure and $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ be a covariant metric on $\mathscr{M}$ satisfying the condition $g\left(X, Y\right) = \gamma\left(X, Y\right)$, $\forall\, X, Y \in \Gamma\left(\text{Ker } \psi\right)$. There is a unique pair $\left(Z, \phi\right)$, with $Z \in FO\left(\mathscr{M}\right)$ a field of observers and $\phi \in C^\infty\left(\mathscr{M}\right)$ a function such that:*

$$g\left(Z\right) = \phi\psi. \tag{3.2.49}$$

Proof: We start by proving that the condition $g\left(X, Y\right) = \gamma\left(X, Y\right)$, $\forall\, X, Y \in \Gamma\left(\text{Ker } \psi\right)$ implies that Rad $g \cap$Ker $\psi = \{0\}$. Suppose there exists a vector field $v \in \Gamma\left(T\mathscr{M}\right)$ such that $g\left(v\right) = \psi\left(v\right) = 0$. Since $\psi\left(v\right) = 0$, $g\left(v, w\right) = \gamma\left(v, w\right) = 0$, $\forall w \in \Gamma\left(\text{Ker } \psi\right)$, which leads to a contradiction since γ is positive definite. In conclusion, such a vector field v does not exist and Rad $g \cap$ Ker $\psi = \{0\}$.

The positive definiteness of γ implies also that the dimension of Rad g is either 0 or 1, so that we will distinguish these two cases:

$\boxed{\text{Dim}\left(\text{Rad } g\right) = 1}$

Let $v \in \Gamma\left(T\mathscr{M}\right)$ such that Rad $g =$ Span v. The defining relation for Z and ϕ then implies $g\left(Z, v\right) = m\,\psi\left(v\right) = 0$, which in turn ensures $\phi = 0$, since $\psi\left(v\right) \neq 0$ in virtue of the precedent discussion. Then, one obtains $g\left(Z\right) = 0$ so that $Z \in$ Rad g, *i.e.* $Z \sim v$. The normalization condition $\psi\left(Z\right) = 1$ fixes Z uniquely.

$\boxed{\text{Dim}\left(\text{Rad } g\right) = 0}$

Since the metric g is now assumed to be nondegenerate, one can introduce its inverse $g^{-1} \in \Gamma\left(\vee^2 T\mathscr{M}\right)$. Acting on each side of the defining equation for Z and ϕ with g^{-1}, one gets $Z = m\,g^{-1}\left(\psi\right)$. Acting now with ψ on each side leads to $\phi\, g^{-1}(\psi, \psi) = 1$, so that $\phi = \frac{1}{g^{-1}(\psi,\psi)}$. Plugging back into the expression for Z leads to $Z = \frac{g^{-1}(\psi)}{g^{-1}(\psi,\psi)}$. We summarise our results in the following table:

Dim (Rad g)	Definition of ϕ	Definition of Z
1	0	$\{Z \in \text{Rad } g, \psi(Z) = 1\}$
0	$\phi = \frac{1}{g^{-1}(\psi,\psi)}$	$Z = \frac{g^{-1}(\psi)}{g^{-1}(\psi,\psi)}$

Note that $\phi = 0$ if and only if Dim (Rad g) $= 1$. $\square$

$\square$

The characterisation of Newtonian manifolds using Coriolis-free fields of observers has thus the nice property to define a (possibly non-degenerate) covariant metric g. Under a Maxwell-gauge transformation $Z \to Z - h\,(df)$, the metric g transforms as

$$g \to g' = g + 2\,\psi \vee df \tag{3.2.50}$$

so that we are led to define an equivalence class $[g]$ as:

Definition 3.2.40 (Lagrangian structure). *Let $\mathscr{L}(\mathscr{M}, \psi, \gamma)$ be a Leibnizian structure. An equivalence class $[g]$ of covariant metrics $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ on $\mathscr{M}$ satisfying the condition $g(X, Y) = \gamma(X, Y)$, $\forall\, X, Y \in \Gamma(\text{Ker } \psi)$ and such that two representatives g' and g differ according to eq.*(3.2.50)*, for some function $f \in C^\infty(\mathscr{M})$, is called a Lagrangian class of metrics. The triplet $\mathcal{L}(\mathscr{M}, \psi, [g])$ is called a Lagrangian structure.*

Now, one can combine Propositions 3.2.36 and 3.2.37 in order to show:

Proposition 3.2.41 (*cf.* [23]). *Let $\mathscr{S}(\mathscr{M}, \psi, \gamma)$ be an Augustinian structure. There is a bijective correspondence between Newtonian manifolds $\mathscr{N}(\mathscr{M}, \psi, \gamma, \nabla)$ and Lagrangian classes of metrics.*

The following table sums up the Milne invariant objects introduced in this Section along with their Maxwell-gauge transformation law:

Type	Name	Definition	Maxwell-gauge transformation law
$Z \in FO(\mathscr{M})$	Coriolis-free field of observer	$Z \equiv N - h\left(\overset{N}{A}\right)$	$Z \to Z - h(df)$
$\phi \in C^{\infty}(\mathscr{M})$	Scalar gravitational potential	$\phi \equiv 2\overset{N}{A}(N) - h\left(\overset{N}{A}, \overset{N}{A}\right)$	$\phi \to \phi + 2\,df\,(Z) - h\,(df, df)$
$g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$	Lagrangian metric	$g \equiv \overset{N}{\gamma} + 2\,\psi \vee \overset{N}{A}$	$g \to g' = g + 2\,\psi \vee df$

Table 3.1: Milne-invariant objects

The denomination Lagrangian metric is justified by the fact that the metric g defines a Lagrangian as $\mathcal{L} \equiv \frac{1}{2} g(X, X)$ with $X \in FO(\mathscr{M})$ the tangent vector field associated to an (arbitrary) observer $x : I \subseteq \mathbb{R} \to \mathscr{M} : \tau \mapsto x(\tau)$. In components, the Lagrangian then reads

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}. \tag{3.2.51}$$

In order to find the associated equations of motion, it must be kept in mind that the variation of the Lagrangian (3.2.51) is not performed over the whole space of tangent vectors but is constrained to the space of tangent vectors parameterised by the proper time τ, *i.e.* to the space of observers (*cf.* Proposition 3.2.8). In the generic case, the constraint $\psi_{\mu} \frac{dx^{\mu}}{d\tau} = 1$ is non-holonomic (*i.e.* of the form $f\left(x^i, \dot{x}^i, t\right) = 0$) and linear in the velocities. However, in the Augustinian case, the absolute clock is closed ($\psi = dt$) so that the constraint can be integrated to give a holonomic constraint (*i.e.* of the form $f\left(x^i, t\right) = 0$) which can be resolved by adopting the absolute time t as parameter:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt}. \tag{3.2.52}$$

The Euler-Lagrange equations of motion derived from $\mathcal{L}$ take the form [23]:

$$g_{\alpha\beta} \frac{d^2 x^{\beta}}{dt^2} + \frac{1}{2}\left[\partial_{\mu} g_{\nu\alpha} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu}\right] \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} = 0.$$

Contracting with $h^{\lambda\alpha}$ and using the relation $g_{\alpha\beta}h^{\lambda\alpha} = \delta^\lambda_\beta - Z^\lambda\psi_\beta$ (as can be deduced from the expression of g) leads to:

$$\frac{d^2x^\lambda}{dt^2} - Z^\lambda\psi_\nu\frac{d^2x^\nu}{dt^2} + \frac{1}{2}h^{\lambda\alpha}\left[\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right]\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0. \tag{3.2.53}$$

Now, differentiating the constraint $\psi_\mu\frac{dx^\mu}{d\tau} = 1$, one obtains the relation

$$\psi_\nu\frac{d^2x^\nu}{dt^2} = -\partial_{(\alpha}\psi_{\beta)}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}$$

which can be substituted in eq.(3.2.53) to give

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0$$

where the components $\Gamma^\lambda_{\mu\nu}$ read

$$\Gamma^\lambda_{\mu\nu} = Z^\lambda\partial_{(\mu}\psi_{\nu)} + \frac{1}{2}h^{\lambda\rho}\left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right]. \tag{3.2.54}$$

Using Table 3.1, one can check that the expression (3.2.54) is identical to the one of eq.(3.2.20), so that the Lagrangian $\mathcal{L}$ describes a free particle in geodesic motion with respect to a Newtonian connection, hence providing a concrete implementation of Proposition 3.2.41. Note that, although being explicitly Milne-invariant, the Lagrangian $\mathcal{L}$ is not invariant under a Maxwell-gauge transformation of g as $g_{\mu\nu} \to g_{\mu\nu} + 2\,\psi_{(\mu}\,\partial_{\nu)}f$ but transforms by adjonction of a total derivative $\mathcal{L} \to \mathcal{L} + \frac{df}{dt}$ which only contributes to the boundary term, so that the equations of motion (and then the expression of $\Gamma^\lambda_{\mu\nu}$) are Maxwell-gauge invariant.

Getting out of the Cave: Towards ambient formalism

> *An unspeakable horror seized me. There was a darkness; then a dizzy, sickening sensation of sight that was not like seeing; I saw a Line that was no Line; Space that was not Space: I was myself, and not myself. When I could find voice, I shrieked aloud in agony, "Either this is madness or it is Hell." "It is neither", calmly replied the voice of the Sphere, "it is Knowledge; it is Three Dimensions: open your eye once again and try to look steadily."*
>
> – Edwin A. Abbott, *Flatland* (1884)

This paragraph is devoted to a heuristic discussion regarding the emergence of the ambient formalism from a nonrelativistic viewpoint, in contradistinction with the top-down approach to be followed in Chapter 4. We here argue that a clever nonrelativistic physicist studying Newtonian manifolds could in fact have a very firm grasp on Bargmann-Eisenhart waves.

Let [10] $\bar{\mathscr{N}}\left(\bar{\mathscr{M}}, \bar{\psi}, \bar{\gamma}, \bar{\nabla}\right)$ be a Newtonian manifold and pick a field of observers $\bar{N} \in FO\left(\bar{\mathscr{M}}\right)$. The characterisation of a Newtonian manifold $\bar{\mathscr{N}}$ has been seen to require the introduction of a set of 1-forms $\overset{\bar{N}}{A} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ with Maxwell-like transformation law $\overset{\bar{N}}{A} \to \overset{\bar{N}}{A} + df$, where $f \in C^\infty\left(\bar{\mathscr{M}}\right)$. To the bundle-minded physicist, this transformation law suggests to reinterpret the 1-forms $\overset{\bar{N}}{A}$ as gauge Ehresmann connections for a principal $(\mathbb{R}, +)$-bundle:

$$\begin{tikzcd} \mathbb{R} \arrow[d, hook] \\ \mathscr{M} \arrow[d, "\pi"] \\ \bar{\mathscr{M}} \end{tikzcd}$$

where $\mathscr{M}$ is a $(d+2)$-dimensional manifold. Recall that, if $\overset{N}{A} \in \Omega^1\left(\mathscr{M}\right)$ is an $(\mathbb{R}, +)$-Ehresmann connection on $\mathscr{M}$, choosing a section $\sigma : \bar{\mathscr{U}} \to \mathscr{M}$ (where $\bar{\mathscr{U}} \subset \bar{\mathscr{M}}$ is an open subset of $\bar{\mathscr{M}}$) allows to define a gauge connection $\overset{\bar{N}}{A} \in \Omega^1\left(\bar{\mathscr{U}}\right)$ as $\overset{\bar{N}}{A} \equiv \sigma^* \overset{N}{A}$. Reciprocally, a collection $\left\{\bar{\mathscr{U}}_\alpha, \overset{\bar{N}}{A}_\alpha\right\}$ (where the $\bar{\mathscr{U}}_\alpha$ form an open cover of $\bar{\mathscr{M}}$ and the set of $\overset{\bar{N}}{A}_\alpha$ differ by Maxwell-like transformation laws) defines an unique Ehresmann connection $\overset{N}{A}$.

The principal $(\mathbb{R}, +)$-bundle involves a supplementary "internal" direction, the vertical fiber foliation, which is a congruence of integral curves for the unique fundamental vector field of the principal bundle $\mathscr{M}$, denoted $\xi \in \Gamma\left(T\mathscr{M}\right)$ and designated as the *wave vector field*. Since ξ is the fundamental vector field, it satisfies $\overset{N}{A}(\xi) = 1$ (since 1 is the generator of the Abelian Lie algebra $\mathbb{R}$).

Usually (*e.g.* in Yang-Mills theories), the fiber of an Ehresmann bundle is interpreted as an auxiliary geometric object allowing to define an internal symmetry. The key to the ambient approach consists in reinterpreting this additional direction as a new spacetime dimension.

10. We anticipate here on the notation to be used in Chapter 4 where nonrelativistic objects are topped with a bar.

Now, we investigate how structures on $\bar{\mathscr{M}}$ can be lifted up to $\mathscr{M}$. First, the absolute clock $\bar{\psi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ defines an unique closed 1-form $\psi \in \Omega^1\left(\mathscr{M}\right)$ as $\psi \equiv \pi^*\bar{\psi}$, called *wave covector field*. It can be checked that $\psi\left(\xi\right) = 0$, so that $\xi \in \operatorname{Ker}\psi$. The kernel of ψ defines an involutive distribution (ψ being closed) whose integral submanifolds are called *wavefront worldvolumes*. Each wavefront worldvolume can thus be envisaged as the union of an absolute space with the fiber. A wavefront wordlvolume $\mathscr{W}$ is therefore endowed with a contravariant metric $\gamma \in \Gamma\left(\vee^2 \operatorname{Ker}\psi\right)$ defined as the generalised pullback of the spatial metric $\bar{\gamma} \in \Gamma\left(\vee^2 \operatorname{Ker}\bar{\psi}\right)$, *i.e.* $\gamma \equiv \pi^*\bar{\gamma}$. Contrarily to its nonrelativistic counterpart, the metric γ is degenerate since $\gamma\left(\xi\right) = 0$ (in the language of [77], the triplet $\left(\mathscr{W}, \gamma, \xi\right)$ is thus a Carroll metric structure). The field of observers $\bar{N} \in FO\left(\bar{\mathscr{M}}\right)$ can be lifted up to $\mathscr{M}$ by defining $N \in FO\left(\mathscr{M}\right)$ as the horizontal lift of $\bar{N}$ with respect to $\overset{N}{A}$ (*i.e.* $\pi_*N = \bar{N}$ and $\overset{N}{A}\left(N\right) = 0$) while an ambient covariant metric $\overset{N}{\gamma} \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ can be defined as the generalised pullback of the transverse metric $\overset{\bar{N}}{\gamma} \in \Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$. It can be checked that $\operatorname{Span}\left\{\xi, N\right\} \in \operatorname{Rad}\overset{N}{\gamma}$.

According to Proposition 3.2.41, a Newtonian manifold defines a class of Lagrangian metrics $[\bar{g}]$ where each metric $\bar{g} \in \Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$ is given by $\bar{g} \equiv \overset{\bar{N}}{\gamma} + 2\,\bar{\psi} \vee \overset{\bar{N}}{A}$ and transforms under a gauge transformation as $\bar{g} \to \bar{g}' = \bar{g} + 2\,\bar{\psi} \vee df$. Similarly to the definition of an Ehresmann connection on $\mathscr{M}$, it can be shown that the set $[\bar{g}]$ defines a unique covariant metric $g \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ as satisfying $\bar{g} = \sigma^* g$. Explicitly, the metric g can be expressed as $g \equiv \overset{N}{\gamma} + 2\,\psi \vee \overset{N}{A}$. Furthermore, the metric g can be shown to be nondegenerate so that $\mathscr{M}$ is a Lorentzian manifold. The expression for g can be used in order to compute $g\left(\xi, \xi\right) = 0$ and $g\left(N, N\right) = 0$ (so that ξ and N are null vector fields). Furthermore, $g\left(\xi\right) = \psi$ and $g\left(N\right) = \overset{N}{A}$. Since g is nondegenerate, it defines a notion of parallelism on $\mathscr{M}$ in the guise of the Levi-Civita connection ∇ and it will be shown in Section 4.4.1 (following [24]) how the Levi-Civita connection ∇ projects downto the Newtonian connection $\bar{\nabla}$ endowing $\bar{\mathscr{M}}$. The wavevector field can then be shown to be parallel with respect to ∇, so that $\mathscr{M}$ can be characterised as a Bargmann-Eisenhart wave.

The conclusion emerging from the line of reasoning sketched here is that the usual hierarchy between Bargmann-Eisenhart waves and Newtonian manifolds (where the latter are obtained from the former) can in fact be reversed given that a deep geometrical understanding of Newtonian manifolds leads naturally to the reconstruction of Bargmann-Eisenhart waves [11].

11. In more Platonic terms, this conclusion can be restated by saying that geometrical understanding provides a way out of the Cave.

3.2.3 Nonrelativistic manifolds based on an Aristotelian structure

Having reviewed in details how a given Augustinian structure can be endowed with a notion of parallelism, we now turn our attention to Aristotelian structures (*cf.* Definition 3.2.15) and investigate how a connection can be defined on such structures. We will propose two competing definitions of connections, both of which will acquire a natural interpretation in the context of the ambient formalism (*cf.* Section 4.4.3 below). Each of these two definitions generalises a different formulation of a Newtonian connection. The first one relies on an alternative set of axioms for a Galilean manifold (as compared with Definition 3.2.18) while the second one adapts the Lagrangian approach to Newtonian connections to the Aristotelian case. This last approach will be seen to be the best suited regarding our aim to geometrise the Eisenhart-Lichnerowicz lift.

Horizontal connection

Before introducing the notion of a Horizontal connection, we first give a set of necessary and sufficient conditions defining a torsionfree Galilean manifold in a form better suited for its subsequent generalization. To this end, we will make use of the notion of conservor:

Definition 3.2.42 (Conservor [23]). *Let ∇ be a Koszul derivative on $\mathscr{M}$ and $\psi \in \Omega^1(\mathscr{M})$ be a 1-form field. The 1-form ψ is said to be a conservor if it satisfies*

$$(\nabla_X \psi)(Y) + (\nabla_Y \psi)(X) = 0, \ \forall\, X, Y \in \Gamma(T\mathscr{M}). \tag{3.2.55}$$

The conservor condition can be seen as an analogue of the Killing equation (*cf.* Definition A.9.8) for non-Riemannian spaces. The terminology, as justified in [23], follows from the Proposition:

Proposition 3.2.43 (*cf.* [23]). *Let $\mathscr{L}(\mathscr{M}, \psi, \gamma)$ be a Leibnizian structure endowed with the Koszul connection ∇ and let $X \in \Gamma(T\mathscr{M})$ be a vector field which is affine geodesic with respect to ∇. If ψ is a conservor for ∇, then the quantity $\psi(X)$ is conserved along the integral curves of X.*

Proof: Starting with the equality $X[\psi(X)] = (\nabla_X \psi)(X) + \psi(\nabla_X X)$. The first term on the right-hand side vanishes due to the conservor condition while the second term is null since X is affine geodesic. Therefore $X[\psi(X)] = 0$, so that the quantity $\psi(X)$ is conserved along X. □

Corollary 3.2.44 (*cf.* [23]). *Let $\mathscr{L}(\mathscr{M}, \psi, \gamma)$ be a Leibnizian structure endowed with the Koszul connection ∇ and assume that ψ is a conservor for ∇. Then any geodesic whose tangent vector lies in the absolute space Σ_t at one point stays entirely in the hypersurface Σ_t.*

Proof: Let $x : I \subseteq \mathbb{R} \to \mathscr{M} : s \mapsto x(s)$ be an integral curve for the vector field X, which is assumed to be affine geodesic. Suppose furthermore that there exists a value $s_0 \in I$ of the curve parameter such that $\psi_{x(s_0)}\left(X_{x(s_0)}\right) = 0$. According to Proposition 3.2.43, $\psi(X) = 0$ is conserved along X so that $\psi_{x(s)}\left(X_{x(s)}\right) = 0$ for all $s \in I$. Therefore, denoting Σ_t the absolute space of absolute time t, $x(s_0) \in \Sigma_t \Rightarrow x(s) \in \Sigma_t$, $\forall s \in I$. $\square$

In the case of a Galilean manifold, the Galilean connection ∇ reduces to the Levi-Civita connection for the spatial metric γ on the absolute spaces Σ_t, so that geodesics whose tangent vector lies in the absolute space Σ_t at one point are geodesics of Σ_t for the spatial Levi-Civita connection. In this particular sense, the absolute spaces of a Galilean manifold can be qualified as totally geodesic.

Equipped with this notion of conservor, we are now able to provide the following alternative definition of torsionfree Galilean manifolds:

Proposition 3.2.45. *Let $\mathscr{S}(\mathscr{M}, \psi, \gamma)$ be an Augustinian structure and ∇ a torsionfree Koszul connection on $\mathscr{M}$. The quadruple $\mathscr{G}(\mathscr{M}, \psi, \gamma, \nabla)$ is a torsionfree Galilean manifold if and only if the Koszul connection ∇ satisfies the following axioms:*

1. *The absolute clock is a conservor for ∇.*
2. *$\exists\ N \in FO(\mathscr{M})\ /\ (\nabla_N h)(\alpha, \beta) = 0$, $\forall \alpha, \beta \in \text{Ann } N$.*
3. *$\nabla_X h = 0$, $\forall X \in \text{Ker } \psi$.*

Proof: We start by showing how the present conditions imply Definition 3.2.18 in the torsionfree case. Since one considers an Augustinian structure, the absolute clock ψ is closed so that $\partial_{[\mu}\psi_{\nu]} = \nabla_{[\mu}\psi_{\nu]} = 0$. Furthermore, ψ is assumed (Axiom 1) to be a conservor with respect to ∇ (*cf.* Definition 3.2.42) so that $\nabla_{(\mu}\psi_{\nu)} = 0$. Both equalities lead to $\nabla_\mu\psi_\nu = 0$ and ∇ is then compatible with ψ (Condition 1 of Definition 3.2.18).

Regarding the compatibility of the contravariant metric h, Axiom 3 implies the existence of a symmetric tensor $S \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ such that $\nabla_\mu h^{\alpha\beta} = \psi_\mu S^{\alpha\beta}$. Furthermore, using Axiom 2, one can decompose the tensor S as $S \equiv N \vee K$, with $K \in \Gamma(T\mathscr{M})$ an arbitrary vector field, so that $\nabla_\mu h^{\alpha\beta} = \psi_\mu N^{(\alpha}K^{\beta)}$. Finally, the consistency requirement $\nabla_\mu \left(h^{\alpha\beta}\psi_\beta\right) = 0$ together with the previously shown compatibility of ψ require that K vanishes, so that ∇ is compatible with h as it should to meet Condition 2 of Definition 3.2.18.

Let us now show how a torsionfree Galilean manifold in the sense of Definition 3.2.18 matches the present definition. As noted earlier, a direct consequence of the compatibility of ∇ with ψ is the fact that only Augustinian structures admit torsionfree compatible Koszul connections (*cf.* Proposition 3.2.20), so that Condition 1 of Definition 3.2.18 ensures that ψ is closed. Furthermore, the same condition ensures Axiom 1. Finally, that Condition 2 of Definition 3.2.18 implies Axioms 2-3 is obvious. □

Note that, although Axiom 2 seems to particularise a field of observers N, the resulting Galilean connection does not depend on a particular N which is consistent with the Milne-invariance of the coefficients (3.2.20).

This new characterisation of torsionfree Galilean manifolds turns out to be better suited in order to endow an Aristotelian structure with a Koszul connection, which is achieved by merely substituting the Augustinian structure of Proposition 3.2.45 by an Aristotelian structure.

Definition 3.2.46 (Horizontal manifold)**.** *Let $\mathscr{A}\left(\mathscr{M},\psi,\gamma\right)$ be an Aristotelian structure and ∇ a torsionfree Koszul connection on $\mathscr{M}$ satisfying Axioms 1-3 of Proposition 3.2.45. The Koszul connection ∇ is then called a Horizontal connection. The quadruple $\mathscr{H}\left(\mathscr{M},\psi,h,\nabla\right)$ is then called a Horizontal manifold.*

The justification for this choice of terminology will be provided in light of the ambient formalism in Section 4.4.3. The following Proposition establishes how compatibility relations of Definition 3.2.18 are affected by the previous substitution:

Proposition 3.2.47. *Let $\mathscr{A}\left(\mathscr{M},\psi,\gamma\right)$ be an Aristotelian structure and ∇ a Horizontal connection on $\mathscr{M}$. Then there exists a field of observers $N\in FO\left(\mathscr{M}\right)$ such that:*

$$\nabla_\mu\psi_\nu = \partial_{[\mu}\ln\Omega\ \psi_{\nu]} \tag{3.2.56}$$

$$\nabla_\mu h^{\alpha\beta} = \psi_\mu N^{(\alpha}h^{\beta)\rho}\partial_\rho\ln\Omega. \tag{3.2.57}$$

where the function $\Omega\in C^\infty\left(\mathscr{M}\right)$ is defined by $d\psi = d\ln\Omega\wedge\psi$.

Proof: First, we recall that since ψ is assumed to induce an involutive distribution, being part of an Aristotelian structure (*cf.* Definition 3.2.15), Frobenius Theorem ensures that locally, there always exists a function $\Omega\in C^\infty\left(\mathscr{M}\right)$ such that $d\psi =$

$d\ln\Omega\wedge\psi$. Eq.(3.2.56) follows straightforwardly from the fact that ψ is assumed to be a conservor (*i.e.* $\nabla_{(\mu}\psi_{\nu)}=0$), so that $\nabla_\mu\psi_\nu=\nabla_{[\mu}\psi_{\nu]}=\partial_{[\mu}\psi_{\nu]}=\partial_{[\mu}\ln\Omega\ \psi_{\nu]}$. Before establishing eq.(3.2.57), let us show how the non-preservation of ψ affects the consistency requirement $\nabla_\mu\left(h^{\alpha\beta}\psi_\beta\right)=0$. Indeed, $\nabla_\mu h^{\alpha\beta}\psi_\beta=-h^{\alpha\beta}\nabla_\mu\psi_\beta=-h^{\alpha\beta}\partial_{[\mu}\ln\Omega\ \psi_{\beta]}$. Now denoting $\nabla_\mu h^{\alpha\beta}\equiv T_\mu^{\alpha\beta}$, the consistency requirement allows to simplify the decomposition of T on a Galilean basis (N,e_i) (with dual $\left(\psi,\theta^i\right)$) as $T_\mu^{\alpha\beta}=T_i^{jk}\theta_\mu^i e_j^{(\alpha}e_k^{\beta)}+T_0^{ij}\psi_\mu e_i^{(\alpha}e_j^{\beta)}+\psi_\mu N^{(\alpha}h^{\beta)\rho}\partial_\rho\ln\Omega$. Axiom 2 imposes furthermore $T_0^{ij}=0$ while Axiom 3 ensures that $T_i^{jk}=0$. □

Similarly to the Galilean case, the "compatibility" conditions eq.(3.2.56)-(3.2.57) do not fix uniquely the torsionfree Koszul connection. A result similar to Theorem 3.2.22 can be formulated as follows:

Proposition 3.2.48. *Given a field of observers $N\in FO(\mathscr{M})$, the set of torsionfree Koszul connections satisfying the relations* (3.2.56)-(3.2.57) *for a given Aristotelian structure $\mathscr{A}(\mathscr{M},\psi,\gamma)$ is in bijective correspondence with the set $\Omega^2(\mathscr{M})$ of 2-forms $\overset{N}{F}$ on $\mathscr{M}$.*

The proof is formally identical to the one of Theorem 3.2.22 and will not be displayed here. This result suggests that, given an Aristotelian structure $\mathscr{A}(\mathscr{M},\psi,\gamma)$, the gift of a couple $\left(N,\overset{N}{F}\right)$ thus determines entirely a "compatible" Koszul connection. Note that contrarily to the Galilean case, there are no equivalence classes involved since changing N amounts to changing the compatibility conditions, and thus ∇. Indeed, one notices that eq.(3.2.57) features the field of observers $N\in FO(\mathscr{M})$ originating from Axiom 2 so that the torsionfree connection thus defined is not independent of N, contrarily to the Augustinian case. We will make explicit this dependence by denoting $\overset{N}{\nabla}$ the Koszul connection associated to a couple $\left(N,\overset{N}{F}\right)$. As usual, the "compatibility" conditions for ψ and h can be used in order to display an explicit expression for the coefficients $\overset{N}{\Gamma}{}^\lambda_{\mu\nu}$ of $\overset{N}{\nabla}$. As it turns out, the obtained coefficients read, in holonomic coordinates

$$\overset{N}{\Gamma}{}^\lambda_{\mu\nu}=N^\lambda\partial_{(\mu}\psi_{\nu)}+\frac{1}{2}h^{\lambda\rho}\left[\partial_\mu\overset{N}{\gamma}_{\rho\nu}+\partial_\nu\overset{N}{\gamma}_{\rho\mu}-\partial_\rho\overset{N}{\gamma}_{\mu\nu}\right]+h^{\lambda\rho}\psi_{(\mu}\overset{N}{F}_{\nu)\rho}\tag{3.2.58}$$

and are therefore formally identical to the ones derived for a torsionfree Galilean connection, with the important difference that the absolute clock ψ is not closed anymore. A major consequence of this fact is that the coefficients (3.2.58) are not invariant under a Milne boost (as is consistent with the dependence on the field of observers N) but rather transforms as $\overset{N}{\Gamma}{}^\lambda_{\mu\nu}\to\overset{N}{\Gamma}{}^\lambda_{\mu\nu}+h^{\lambda\alpha}\partial_{[\alpha}\psi_{\nu]}\Phi_\mu+h^{\lambda\alpha}\partial_{[\alpha}\psi_{\mu]}\Phi_\nu$, with Φ as in (3.2.18). This transformation relation shows explicitly how the dependence on N can be related to the fact that ψ fails to be closed.

Platonic manifolds

In the present Section, we broach a different notion of parallelism that can endow Aristotelian structures and hence provide the necessary tools in order to apprehend the geometry underlying the Eisenhart-Lichnerowicz lift. This is more naturally done by generalising the Lagrangian approach to Newtonian connections to the Aristotelian case. Our starting point is thus an Aristotelian structure $\mathscr{A}(\mathscr{M}, \psi, \gamma)$ endowed with a Lagrangian metric $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$. We write the Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.2.59}$$

where $x : I \subseteq \mathbb{R} \to \mathscr{M} : \tau \mapsto x(\tau)$ is an observer.

Note that, in contradistinction with the Augustinian case, the absolute clock ψ is not closed, and the normalisation condition $\psi_\mu \frac{dx^\mu}{d\tau} = 1$ is therefore a non-holonomic constraint. Taking this constraint into account while varying the action

$$\mathcal{S} = \int \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau \tag{3.2.60}$$

we find the following Euler-Lagrange equations of motion:

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{3.2.61}$$

where the coefficients $\Gamma^\lambda_{\mu\nu}$ now read:

$$\Gamma^\lambda_{\mu\nu} = Z^\lambda \partial_{(\mu}\psi_{\nu)} + \frac{1}{2} h^{\lambda\rho}\left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right] + g_{\mu\nu} h^{\lambda\rho} \partial_{[\rho}\psi_{\alpha]} Z^\alpha. \tag{3.2.62}$$

Comparing with the same derivation in the Augustinian case, we note the appearance of an extra term $g_{\mu\nu} h^{\lambda\rho} \partial_{[\rho}\psi_{\alpha]} Z^\alpha$ whose presence is due to the non-holonomicity of the constraint $\psi_\mu \frac{dx^\mu}{d\tau} = 1$. It can be checked that expression (3.2.62) is invariant (when the absolute clock satisfies the Frobenius Criterion) under a Maxwell-gauge transformation (*cf.* Table 3.1). Similarly to the Newtonian case, one can make use of the relations of Table 3.1 in order to recast the previous expression in the form:

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= N^\lambda\partial_{(\mu}\psi_{\nu)} + \frac{1}{2}h^{\lambda\rho}\left[\partial_\mu\overset{N}{\gamma}_{\rho\nu} + \partial_\nu\overset{N}{\gamma}_{\rho\mu} - \partial_\rho\overset{N}{\gamma}_{\mu\nu}\right] + h^{\lambda\rho}\psi_{(\mu}\overset{N}{F}_{\nu)\rho} \\ &\quad + \overset{N}{\gamma}_{\mu\nu}h^{\lambda\rho}\partial_{[\rho}\psi_{\alpha]}N^\alpha \end{aligned} \tag{3.2.63}$$

where the 2-form $\overset{N}{F} \equiv d\overset{N}{A} \in \Omega^2(\mathscr{M})$ is exact. Comparison with the coefficients (3.2.58) reveals the presence of the extra term $\overset{N}{\gamma}_{\mu\nu}h^{\lambda\rho}\partial_{[\rho}\psi_{\alpha]}N^\alpha$. This extra term has the nice feature of making coefficients (3.2.63) Milne-invariant (again, when the absolute clock ψ satisfies the Frobenius Criterion). Thus no field of observers acquires a privileged status, in contradistinction with the preferred field of observers characteristic of the Horizontal connection.

However, a peculiarity of the present connection lies in the fact that it does not reduce to the Levi-Civita connection for the spatial metric γ on the absolute spaces Σ_t. Rather, the extra term $\overset{N}{\gamma}_{\mu\nu}h^{\lambda\rho}\partial_{[\rho}\psi_{\alpha]}N^\alpha$ furnishes a contribution so that

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left[\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk} + \gamma_{jk}\partial_l \ln\Omega\right].$$

This odd feature is in contradistinction with both Galilean connections and Horizontal connections but will appear as a necessary evil in order to acquire a geometric understanding of the Eisenhart-Lichnerowicz lift.

As a first step, we will rely on the coefficients (3.2.63) in order to define the Aristotelian analogue of a Newtonian connection, dubbed Platonic connection:

Definition 3.2.49 (Platonic manifold)**.** *Let $\mathscr{A}(\mathscr{M},\psi,\gamma)$ be an Aristotelian structure endowed with an equivalence class $\left[N,\overset{N}{A}\right]$ defined as in eq.*(3.2.44)*. The Koszul connection whose coefficients are given by eq.*(3.2.63) *is called a Platonic connection. The quadruple $\mathscr{P}(\mathscr{M},\psi,\gamma,\nabla)$ is then called a Platonic manifold.*

Progressing towards more intrinsic characterisations of Platonic connections, we now investigate in what sense a Platonic manifold can be said to be conformally related to a Newtonian manifold. The two next definitions provide a meaning to this notion at the level of structures (*cf.* Definition 3.2.50) and manifolds (*cf.* Definition 3.2.51).

By definition of an Aristotelian structure, the absolute clock ψ satisfies the Frobenius Criterion and thus there exists a closed 1-form $\bar{\psi}$ and a positive function Ω (called time unit of ψ) such that $\psi = \Omega\bar{\psi}$. Therefore, one can formulate the following Definition:

Definition 3.2.50 (Conformally related structures)**.** *An Augustinian structure $\mathscr{S}\left(\mathscr{M},\bar{\psi},\bar{\gamma}\right)$ and an Aristotelian structure $\mathscr{A}(\mathscr{M},\psi,\gamma)$ are said to be conformally related if the following* *relations hold:*

$$\begin{cases} \psi = \Omega\,\bar{\psi} \\ \gamma = \Omega\,\bar{\gamma} \\ h = \Omega^{-1}\bar{h} \end{cases} \tag{3.2.64}$$

with $\Omega \in C^\infty(\mathscr{M})$ the time unit of ψ.

Definition 3.2.51 (Conformally related manifolds)**.** *Let $\mathscr{A}(\mathscr{M},\psi,\gamma)$ be an Aristotelian structure conformally related to the Augustinian structure $\mathscr{S}\left(\mathscr{M},\bar{\psi},\bar{\gamma}\right)$. Let ∇ be a Platonic connection endowing $\mathscr{A}$ associated to the equivalence class $\left[N,\overset{N}{A}\right]$ and $\bar{\nabla}$ a Newtonian connection endowing $\mathscr{S}$ associated to the equivalence class $\left[\bar{N},\overset{N}{A}\right]$. The Platonic manifold $\mathscr{P}(\mathscr{M},\psi,\gamma,\nabla)$ is said to be conformally related to the Newtonian manifold $\mathscr{N}\left(\mathscr{M},\bar{\psi},\bar{\gamma},\bar{\nabla}\right)$ if $N=\Omega^{-1}\bar{N}$, with $\Omega \in C^\infty(\mathscr{M})$ the time unit of ψ .*

Proposition 3.2.52. *Let $\mathscr{A}(\mathscr{M},\psi,\gamma)$ be an Aristotelian structure conformally related to the Augustinian structure $\mathscr{S}\left(\mathscr{M},\bar{\psi},\bar{\gamma}\right)$ and let ∇ be a Koszul connection endowing $\mathscr{M}$. The Koszul connection ∇ is a Platonic connection for $\mathscr{A}$ if and only if there exists a Newtonian connection $\bar{\nabla}$ endowing $\mathscr{S}$ such that*

$$\forall X,\,\bar{X} \in \Gamma(T\mathscr{M}) \ / \ X = \Omega^{-1}\bar{X}, \quad \nabla_X X = \Omega^{-2}\bar{\nabla}_{\bar{X}}\bar{X}. \tag{3.2.65}$$

Proof: Let $\bar{\nabla}$ be a Newtonian connection associated to the equivalence class $\left[\bar{N},\overset{N}{A}\right]$.

The coefficients $\bar{\Gamma}^\lambda_{\mu\nu}$ are given by eq.(3.2.20), with $\overset{N}{F} \equiv d\overset{N}{A}$. Computing $\Omega^{-2}\bar{\nabla}_{\bar{X}}\bar{X}$ and using $X=\Omega^{-1}\bar{X}$ leads to

$$\Omega^{-2}\bar{\nabla}_{\bar{X}}\bar{X}^\lambda = X^\mu\partial_\mu X^\lambda + \Gamma^\lambda_{\mu\nu}X^\mu X^\nu \tag{3.2.66}$$

where the coefficients $\Gamma^\lambda_{\mu\nu}$ are given by (3.2.63) with

$$\begin{cases} \psi = \Omega\,\bar{\psi} \\ h = \Omega^{-1}\bar{h} \\ N = \Omega^{-1}\bar{N} \\ \overset{N}{\gamma} = \Omega\,\bar{\gamma}. \end{cases} \tag{3.2.67}$$

Since relation (3.2.66) holds for all $X \in \Gamma(T\mathscr{M})$, then ∇ is the Platonic connection conformally related to $\bar{\nabla}$. The converse statement is straightforward by the same computation. □

Condition (3.2.65) is reminiscent of the equation relating the affine parameterisations of a null geodesic vector field in two conformally related Riemannian manifolds (*cf.* *e.g.* Appendix D of [90]). Indeed, in Section (4.4.5), we will reintepret this condition along these lines using hindsight provided by the ambient approach.

Definitions 3.2.49 and Proposition 3.2.52 can be easily generalised in order to characterise a "Galilean" equivalent to a Platonic connection, *i.e.* a connection with coefficients given by (3.2.63) with $\overset{N}{F}$ non-necessarily closed. We conclude the present Section by prosing an axiomatic formulation for such a "Galilean" equivalent of a Platonic connection mimicking Definition 3.2.18.

From the expression (3.2.63) for the coefficients of a Platonic connection, we can compute the "compatibility" conditions of ∇ with ψ and h: [12]

$$\begin{aligned} \nabla_\mu \psi_\nu &= \partial_{[\mu} \ln \Omega \ \psi_{\nu]} && (3.2.68) \\ \nabla_\mu h^{\alpha\beta} &= \delta_\mu^{(\alpha} h^{\beta)\rho} \partial_\rho \ln \Omega. && (3.2.69) \end{aligned}$$

Adopting a bottom-top approach, we can formulate a set of axioms allowing to recover the compatibility conditions (3.2.68)-(3.2.69) as follows:

Proposition 3.2.53. *Let $\mathscr{A}(\mathscr{M}, \psi, \gamma)$ be an Aristotelian structure and ∇ a torsionfree Koszul connection on $\mathscr{M}$. The compatibility conditions* (3.2.68)-(3.2.69) *are satisfied if and only if ∇ satisfies the following axioms:*

1. *The absolute clock is a conservor for ∇.*
2. *$\exists\ N \in FO(\mathscr{M})$ /*
   (a) *$(\nabla_N h)(\alpha, \beta) = 0$, $\forall \alpha, \beta \in \text{Ann } N$.*
   (b) *$\exists$ a vector field $V \in \Gamma(T\mathscr{M})$ / $\nabla_N h^{\alpha\beta} = N^{(\alpha} V^{\beta)}$.*

Proof: Making use of the arguments composing the proof of Proposition 3.2.47 ensures that a torsionfree Koszul connection satisfying Axiom 1 admits eq.(3.2.68) as compatibility conditions for the absolute clock. Furthermore, we showed there that, denoting $\nabla_\mu h^{\alpha\beta} \equiv T_\mu^{\alpha\beta}$, the decomposition of T on a Galilean basis (N, e_i) (with dual (ψ, θ^i)) reads $T_\mu^{\alpha\beta} = T_i^{jk} \theta_\mu^i e_j^{(\alpha} e_k^{\beta)} + \psi_\mu N^{(\alpha} h^{\beta)\rho} \partial_\rho \ln \Omega$, where Axiom 2.($a$) has been used. Now, Axiom 2.(b) guarantees that $\nabla_N h^{\alpha\beta} = N^\mu T_\mu^{\alpha\beta} = N^{(\alpha} h^{\beta)\rho} \partial_\rho \ln \Omega = N^{(\alpha} V^{\beta)}$. Contracting with ψ_α leads to $h^{\beta\rho} \partial_\rho \ln \Omega = V^\beta + N^\beta V^0$. Contracting again with ψ_β

12. Again, comparing the relation (3.2.69) with eq.(3.2.57) reveals the absence of a privileged field of observers N which characterised the connection $\overset{N}{\nabla}$.

imposes $V^0 = 0$, so that $V^\beta = h^{\beta\rho}\partial_\rho \ln \Omega$, and compatibility condition (3.2.69) is met. $\square$

Chapter 4

Ambient geometric structures: Gravitational waves as principal bundles

The ambition of the present Chapter is to provide a complete account regarding the embedding of nonrelativistic structures inside gravitational waves. In order to do so, we will rely on the language of principal bundles, in which we start by reformulating the notions regarding gravitational waves introduced in Chapter 2 (Section 4.1). In Section 4.2, the results obtained in Chapter 2 regarding the embedding of nonrelativistic metric structures inside Platonic waves are generalised to the larger class of Kundt waves. Section 4.3 will act as an interlude allowing us to explore the relativistic avatar of field of observers as well as some related notions such as Bargmann bases and Ehresmann connections. Section 4.4 will address the notion of parallelism in the ambient formalism, thus providing a detailed account regarding the projection of the Levi-Civita connection of a gravitational wave. We will start by reviewing the construction of [24] for Bargmann-Eisenhart waves (Section 4.4.1) and then investigate some generalisations to larger class of gravitational waves. As a first step, we will pursue the generalisation initiated in Section 4.2 by showing how the Levi-Civita connection of a Kundt wave projects onto the nonrelativistic absolute spaces. This preliminary step will prepare the subsequent generalisation of the construction of [24] to the class of Platonic waves (*cf.* Section 4.4.3). The most striking feature of this generalisation lies in the non-uniqueness of the induced nonrelativistic Koszul connection. We will discuss two different prescriptions, allowing to recover the Horizontal and Platonic connections introduced in the previous Chapter. Finally, we will conclude by making use of these new notions of parallelism in order to provide a geometric formulation of the Eisenhart-Lichnerowicz Theorem.

4.1 General setup

Let $(\mathscr{M}, g, \xi)$ be a gravitational wave (*cf.* Definition 2.1.1) with $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ a covariant metric and $\xi \in \Gamma(T\mathscr{M})$ the *wave vector field*[1]. The *wave covector field* associated to ξ will be denoted $\psi \equiv g(\xi) \in \Omega^1(\mathscr{M})$. By definition, ξ is hypersurface-orthogonal so that the kernel of the 1-form ψ induces an involutive distribution $\mathcal{D} = \{\mathcal{D}_x\}$, with $\mathcal{D}_x \equiv \text{Ker } \psi_x$. The integral submanifolds of the distribution $\mathcal{D}$ are called *wavefront worldvolumes* and, due to the involutivity of $\mathcal{D}$, constitute a foliation of $\mathscr{M}$.

We now define an integral curve $\gamma_x : \mathbb{R} \to \mathscr{M} : \lambda \mapsto \gamma_x(\lambda)$ of the wave vector field as the unique solution to the differential equation [2]

$$\frac{d\gamma_x(\lambda)}{d\lambda} = \xi_{\gamma_x(\lambda)}$$

with $\lambda \in \mathbb{R}$ and initial condition $\gamma_x(0) = x$. Since ξ is assumed to be complete, its integral curves exist for all values of the parameter λ and, consequently, the flow

$$\varphi_\xi : \mathscr{M} \times \mathbb{R} \to \mathscr{M} : (x, \lambda) \mapsto \gamma_x(\lambda)$$

is global and therefore induces a well-defined right-action of the additive Lie group $(\mathbb{R}, +)$ on $\mathscr{M}$. Integral curves $I_x = \{\gamma_x(\lambda) \in \mathscr{M} \,|\, \lambda \in \mathbb{R}\}$ for $x \in \mathscr{M}$ are therefore orbits of the $(\mathbb{R}, +)$-action φ_ξ through x.

The *Platonic screen* is then defined as the orbit space of the action φ_ξ *i.e.* the set $\bar{\mathscr{M}} \equiv \mathscr{M}/\mathbb{R} = \{I_x \,|\, x \in \mathscr{M}\}$ of all integral curves on $\mathscr{M}$. The following Proposition gives sufficient conditions for the Platonic screen to be a smooth manifold:

Proposition 4.1.1. *The Platonic screen of a gravitational wave whose wave vector ξ induces a free and proper action via its flow φ_ξ is a smooth manifold.*

Proof: This Proposition is a direct application of the quotient manifold Theorem (A.6.2). □

In the following, we will place ourselves in the conditions of application of Proposition 4.1.1. The quotient manifold Theorem ensures furthermore that the projection map onto orbits of φ_ξ, denoted $\pi : \mathscr{M} \to \bar{\mathscr{M}} : x \mapsto I_x$ is a submersion and therefore defines the

1. In the following, the wave vector field ξ will be assumed to be affine geodesic, *cf.* Proposition 2.1.4.
2. This differential equation can be alternatively formulated as $\gamma_{x*}D_0 = \xi$, with $D_s = \frac{\partial}{\partial t}|_{t=s}$.

principal fiber bundle:

$$\begin{tikzcd} \mathbb{R} \arrow[d, hook] \\ \mathscr{M} \arrow[d, "\pi"] \\ \bar{\mathscr{M}} \end{tikzcd} \tag{4.1.1}$$

whose fibers are the integral curves I_x. The vertical subspace $V_x \in T_x\mathscr{M}$ at a point $x \in \mathscr{M}$ is therefore spanned by ξ_x, *i.e.* $V_x = \text{Span}\ \xi_x$, so that $\pi_*\xi = 0$ where $\pi_* : T\mathscr{M} \to T\bar{\mathscr{M}}$ designates the pushforward of the map π.

According to Proposition A.6.7, the principal bundle $\mathscr{M}$ can be made trivial, so that $\mathscr{M}$ is in fact isomorphic to $\bar{\mathscr{M}} \times \mathbb{R}$. The triviality of $\mathscr{M}$ ensures the existence of global cross sections (*cf.* Proposition A.6.6) which are in one-to-one correspondence with *screen worldvolumes.*

Tangent spaces

This Section aims at characterising the tangent spaces of the Platonic screen $\bar{\mathscr{M}}$. We start by introducing the equivalence relation $\sim$ and then prove two Propositions relating $T\bar{\mathscr{M}}$ and $T^*\bar{\mathscr{M}}$ to $T\mathscr{M}/\sim$ and $\text{Ann}\ \xi$ respectively:

Definition 4.1.2 (Equivalence relation $\sim$)**.** *At each point $x \in \mathscr{M}$, two vectors X_x, $Y_x \in T_x\mathscr{M}$ are said equivalent by $\sim$ (i.e. $X_x \sim_x Y_x$) if and only if the following Proposition holds*

$$\exists\, \alpha \in \mathbb{R} / X_x = Y_x + \alpha\, \xi_x.$$

The equivalence classes are denoted $\bar{X}_x$ and the set of equivalence classes is the quotient set denoted $T_x\mathscr{M}/\sim_x$. We denote P the projection map $P : T_x\mathscr{M} \to T_x\mathscr{M}/\sim_x : X_x \mapsto \bar{X}_x$.

Proposition 4.1.3. *For all $x \in \mathscr{M}$, there exists a canonical isomorphism $T_{\pi(x)}\bar{\mathscr{M}} \simeq T_x\mathscr{M}/\sim_x$.*

Proof: One starts by noting that $X_x \sim_x Y_x$ if and only if $\pi_*(X_x) = \pi_*(Y_x)$ due to the fact that ξ is vertical. Thus, Proposition A.1.3 ensures that there exists a unique continuous map $g : T_x\mathscr{M}/\sim_x \to T_{\pi(x)}\bar{\mathscr{M}}$ such that $\pi_* = g \circ P$. Furthermore, π being a submersion implies that π_* is surjective, therefore g is surjective. Plus, P being a projection, is also surjective. Let us prove that g is also injective. If

g is not injective, then $\exists\ \bar{X}_x^1, \bar{X}_x^2 \in T_x\mathscr{M}/\sim_x$ such that $\bar{X}_x^1 \neq \bar{X}_x^2$ and $g\left(\bar{X}_x^1\right) = g\left(\bar{X}_x^2\right)$. Since P is surjective, we have $g\left(P\left(X_x^1\right)\right) = g\left(P\left(X_x^2\right)\right)$ with $X_x^1, X_x^2 \in T_x\mathscr{M}$. Therefore, $\pi_*\left(X_x^1\right) = \pi_*\left(X_x^2\right)$ which implies $X_x^1 \sim_x X_x^2$ and then $P\left(X_x^1\right) = P\left(X_x^2\right)$ and $\bar{X}_x^1 = \bar{X}_x^2$ which leads to a contradiction, therefore g is injective. Being surjective and injective, g is bijective. We now show that g is linear. Let $\bar{X}_x^1, \bar{X}_x^2 \in T_x\mathscr{M}/\sim_x$ and $\lambda \in \mathbb{R}$, $g\left(\lambda\bar{X}_x^1 + \bar{X}_x^2\right) = g\left(\lambda P\left(X_x^1\right) + P\left(X_x^2\right)\right)$ with $X_x^1, X_x^2 \in T_x\mathscr{M}$ and where we have used the fact that P is surjective. Now the linearity of P implies $g\left(\lambda\bar{X}_x^1 + \bar{X}_x^2\right) = g\left(P\left(\lambda X_x^1 + X_x^2\right)\right) = \pi_*\left(\lambda X_x^1 + X_x^2\right)$ and using the linearity of π_* leads to $g\left(\lambda\bar{X}_x^1 + \bar{X}_x^2\right) = \lambda\pi_*\left(X_x^1\right) + \pi_*\left(X_x^2\right) = \lambda g\left(P\left(X_x^1\right)\right) + g\left(P\left(X_x^2\right)\right) = \lambda g\left(\bar{X}_x^1\right) + g\left(\bar{X}_x^2\right)$, therefore g is linear. The linear bijective map g then defines a canonical isomorphism between $T_{\pi(x)}\bar{\mathscr{M}}$ and $T_x\mathscr{M}/\sim_x$, $\forall x \in \mathscr{M}$. $\square$

Proposition 4.1.4. *The pullback π^* defines a canonical isomorphism between $T^*_{\pi(x)}\bar{\mathscr{M}}$ and* $\text{Ann}\ \xi_x$ *the annihilator of* $\text{Span}\ \xi_x$ *in $T^*_x\mathscr{M}$, i.e. $\pi^* : T^*_{\pi(x)}\bar{\mathscr{M}} \to$* $\text{Ann}\ \xi_x$ *is an isomorphism, $\forall x \in \mathscr{M}$.*

Proof: The vector spaces $T^*_{\pi(x)}\bar{\mathscr{M}}$ and $\text{Ann}\ \xi_x$ have same (finite) dimension, and are therefore isomorphic. The pullback $\pi^* : T^*_{\pi(x)}\bar{\mathscr{M}} \to T^*_x\mathscr{M}$ maps elements of $T^*_{\pi(x)}\bar{\mathscr{M}}$ to elements of $\text{Ann}\ \xi_x$, as can be seen by computing, for $\bar{\psi}_{\pi(x)} \in T^*_{\pi(x)}\bar{\mathscr{M}}$, $\pi^*\left(\bar{\psi}_{\pi(x)}\right)(\xi_x) = \bar{\psi}_{\pi(x)}\left(\pi_*\left(\xi_x\right)\right) = 0$, $\forall\bar{\psi}_{\pi(p)} \in T^*_{\pi(x)}\bar{\mathscr{M}}$. Therefore $\text{Im}\left(\pi^*\right) \subset \text{Ann}\ \xi_x$. Furthermore, $\pi^* : T^*_{\pi(x)}\bar{\mathscr{M}} \to \text{Ann}\ \xi_x$ being injective (*cf.* Proposition A.2.10) is also bijective, as follows from the fact that $T^*_{\pi(x)}\bar{\mathscr{M}}$ and $\text{Ann}\ \xi_x$ are isomorphic. Finally, the linearity of π^* implies that $\pi^* : T^*_{\pi(x)}\bar{\mathscr{M}} \to \text{Ann}\ \xi_x$ is an isomorphism. $\square$

The following three Sections intend to make use of the previous characterisation of tangent spaces of $\bar{\mathscr{M}}$ in order to discriminate among the fields living on the gravitational wave $(\mathscr{M}, g, \xi)$ those admitting a well-defined projection on the Platonic screen $\bar{\mathscr{M}}$.

Projection of a function

Definition 4.1.5 (ξ-invariant function). *A function $f \in C^\infty(\mathscr{M})$ on the gravitational wave $\mathscr{M}$ is said ξ-invariant if it satisfies the relation $\mathcal{L}_\xi f = 0$.*

Definition 4.1.6 (Lift of a function). *Let $\bar{f} \in C^\infty\left(\bar{\mathscr{M}}\right)$ be a function on the Platonic screen $\bar{\mathscr{M}}$. The function $\bar{f}$ defines uniquely a function $f \equiv \bar{f} \circ \pi \in C^\infty(\mathscr{M})$ on the gravitational wave $\mathscr{M}$, called the lift of $\bar{f}$.*

Proposition 4.1.7. *There is a bijective correspondence between ξ-invariant functions on the gravitational wave $\mathscr{M}$ and functions on the Platonic screen $\bar{\mathscr{M}}$.*

Proof: Let $\bar{f} \in C^\infty\left(\bar{\mathscr{M}}\right)$ be a function on the Platonic screen $\bar{\mathscr{M}}$. The lifted function $f \equiv \bar{f} \circ \pi \in C^\infty\left(\mathscr{M}\right)$ is ξ-invariant since $\mathcal{L}_\xi f = \xi\left[f\right] = \xi\left[\bar{f} \circ \pi\right] = \left(\pi_* \xi\right)\left[\bar{f}\right] \circ \pi = 0$ where relation (A.2.4) and the verticality of ξ have been used.

Conversely, any ξ-invariant function $f \in C^\infty\left(\mathscr{M}\right)$ induces a well-defined function $\bar{f} \equiv f \circ \sigma$ on the Platonic screen, with $\sigma : \bar{\mathscr{M}} \to \mathscr{M}$ a section. The equivariance of f guarantees that $\bar{f}$ is independent of the choice of section.

□

Projection of a vector field

Definition 4.1.8 (Projectable and ξ-invariant vector fields)**.** *Let $\left(\mathscr{M}, g, \xi\right)$ be a gravitational wave. The vector field $X \in \Gamma\left(T\mathscr{M}\right)$ on $\mathscr{M}$ is said projectable (respectively ξ-invariant) if it satisfies $\mathcal{L}_\xi X = f\xi$, for some function $f \in C^\infty\left(\mathscr{M}\right)$ (resp. $\mathcal{L}_\xi X = 0$).*

Proposition 4.1.9. *If X and $Y \in \Gamma\left(T\mathscr{M}\right)$ are two projectable (resp. ξ-invariant) vector fields on $\mathscr{M}$, then their Lie bracket $\left[X, Y\right] \in \Gamma\left(T\mathscr{M}\right)$ is projectable (resp. ξ-invariant).*

Proof: Let $f, g \in C^\infty\left(\mathscr{M}\right)$ be two functions of $\mathscr{M}$ satisfying respectively $\left[\xi, X\right] = f\xi$ and $\left[\xi, Y\right] = g\xi$. Using Jacobi identity, one writes:

$$\begin{aligned}
\left[\left[X, Y\right], \xi\right] &= -\left[\left[\xi, X\right], Y\right] - \left[\left[Y, \xi\right], X\right] \\
&= -f\left[\xi, Y\right] + Y\left[f\right]\xi + g\left[\xi, X\right] - X\left[g\right]\xi \\
&= -\left(X\left[g\right] - Y\left[f\right]\right)\xi
\end{aligned}$$

so that $\left[X, Y\right]$ is projectable. The ξ-invariant case follows straightforwardly by putting $f = g = 0$. □

Definition 4.1.10 ((ξ-invariant)-Lift)**.** *Let $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ be a vector field on the Platonic screen $\bar{\mathscr{M}}$. A vector field $X \in \Gamma\left(T\mathscr{M}\right)$ satisfying the conditions:*

- *X is projectable (resp. ξ-invariant)*
- *$\pi_* X = \bar{X}$*

is called a lift (resp. ξ-invariant lift) of $\bar{X}$ in $\mathscr{M}$.

Proposition 4.1.11. *Let $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ be a vector field on the Platonic screen $\bar{\mathscr{M}}$. Let X and $X' \in \Gamma\left(T\mathscr{M}\right)$ be two lifts (resp. ξ-invariant lifts) of $\bar{X}$ in $\mathscr{M}$. Then, there exists a (resp. ξ-invariant) function $f \in C^\infty\left(\mathscr{M}\right)$ such that $X' = X + f\xi$.*

Proof: Since X and X' project onto the same vector field $\bar{X}$, they must differ by a vertical vector field *i.e.* there exists a function $f \in C^\infty(\mathscr{M})$ such that $X' = X + f\xi$. In the case of two ξ-invariant lifts, the supplementary condition $[\xi, X'] = [\xi, X] = 0$ imposes $\mathcal{L}_\xi f = 0$. $\square$

Proposition 4.1.12. *Let $\bar{X}$ and $\bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ be two vector fields on the Platonic screen $\bar{\mathscr{M}}$ and let $X \in \Gamma(T\mathscr{M})$ (resp. $Y \in \Gamma(T\mathscr{M})$) designates a lift of $\bar{X}$ (resp. $\bar{Y}$). The following equality stands:*

$$\pi_*[X, Y] = \left[\bar{X}, \bar{Y}\right]$$

where $[\cdot, \cdot]$ designates the Lie bracket.

Proof: Proposition 4.1.9 ensures that the Lie bracket $[X, Y]$ is projectable, so that the vector field $\pi_*[X, Y] \in \Gamma\left(T\bar{\mathscr{M}}\right)$ is well-defined. Since the pushforward operation commutes with the Lie bracket, one obtains $\pi_*[X, Y] = [\pi_* X, \pi_* Y] = \left[\bar{X}, \bar{Y}\right]$. $\square$

Lemma 4.1.13. *Let $(\mathscr{M}, g, \xi)$ be a gravitational wave with Platonic screen $\bar{\mathscr{M}}$. Let $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ be a vector field and $\bar{f} \in C^\infty\left(\bar{\mathscr{M}}\right)$ be a function on $\bar{\mathscr{M}}$. Let $X \in \Gamma(T\mathscr{M})$ be a lift of $\bar{X}$. Then the vector field $fX \in \Gamma(T\mathscr{M})$ is a lift of $\bar{f}\bar{X}$, where $f \in C^\infty(\mathscr{M})$ is the lift of $\bar{f}$ on $\mathscr{M}$ (cf. Proposition 4.1.7). Furthermore, if X is ξ-invariant, so is fX.*

Proof: We need to prove that $\pi_*(fX) = \bar{f}\bar{X}$. Introducing a function $\bar{g} \in C^\infty\left(\bar{\mathscr{M}}\right)$ and using eq.(A.2.4) leads to $(\pi_*(fX))[g] \circ \pi = fX[g \circ \pi]$. Straightforward manipulations lead to:

$$\begin{aligned}
(\pi_*(fX))[\bar{g}] \circ \pi &= fX[\bar{g} \circ \pi] \\
&= f \cdot (\pi_* X)[\bar{g}] \circ \pi \\
&= \left(\bar{f} \circ \pi\right) \bar{X}[\bar{g}] \circ \pi \\
&= \left(\bar{f}\bar{X}[\bar{g}]\right) \circ \pi.
\end{aligned}$$

This expression stand for all functions $\bar{g}$, so that $\pi_*(fX) = \bar{f}\bar{X}$. The function $f \equiv \bar{f} \circ \pi$ is ξ-invariant by construction, so that fX is ξ-invariant if X is. $\square$

Projection of a 1-form

From Proposition 4.1.4, it follows that a necessary condition in order for a 1-form $\alpha \in \Omega^1(\mathscr{M})$ to project onto a well-defined 1-form $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ is to satisfy $\alpha(\xi) = 0$. Now, in the same fashion that integral curves may be defined as paths obtained by fixing a point $x \in \mathscr{M}$ and acting with φ_ξ, *i.e.* $\gamma_x(\lambda) = \varphi_\xi(x, \lambda)$, we can also choose to fix an element $\lambda \in \mathbb{R}$ and define the application $L_\lambda : \mathscr{M} \to \mathscr{M} : x \mapsto L_\lambda(x) \equiv \varphi_\xi(\lambda, x) = \gamma_x(\lambda)$. The regularity of the action φ_ξ on each orbit ensures that L_λ maps bijectively each fiber I_x into itself. Since we only consider smooth functions, $L_\lambda : I_x \to I_x$ is a diffeomorphism. Let us denote $\pi^*_{L_\lambda(x)}$ the pullback $\pi^*_{L_\lambda(x)} : T^*_{\pi(x)}\bar{\mathscr{M}} \to \text{Ann}\ \xi_{L_\lambda(x)}$ which assigns to a 1-form $\bar{\alpha}_{\pi(x)} \in T^*_{\pi(x)}\bar{\mathscr{M}}$ a 1-form $\alpha_{L_\lambda(x)} \in \text{Ann}\ \xi_{L_\lambda(x)}$ at every point of the orbit I_x. Explicitly, the 1-forms pulled back from $\bar{\alpha}_{\pi(x)}$ at x and $L_\lambda(x)$ read $\alpha_x \equiv \pi^*_x\bar{\alpha}_{\pi(x)}$ and $\alpha_{L_\lambda(x)} \equiv \pi^*_{L_\lambda(x)}\bar{\alpha}_{\pi(x)}$. Acting on $\alpha_{L_\lambda(x)}$ with L^*_λ, we have:

$$\begin{aligned}
L^*_\lambda \alpha_{L_\lambda(x)} &= L^*_\lambda \circ \pi^*_{L_\lambda(x)}\bar{\alpha}_{\pi(x)} \\
&= \left(\pi_{L_\lambda(x)} \circ L_\lambda\right)^* \bar{\alpha}_{\pi(x)} \\
&= \pi^*_x \bar{\alpha}_{\pi(x)} \\
&= \alpha_x
\end{aligned}$$

where the composition law of pullbacks has been used. A necessary condition for the projection of a 1-form $\alpha \in \Omega^1(\mathscr{M})$ onto $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ to be well-defined is then that $L^*_\lambda \alpha_{L_\lambda(x)} = \alpha_x$, $\forall x \in \mathscr{M}, \lambda \in \mathbb{R}$. This condition is equivalent to $\mathcal{L}_\xi \alpha = 0$ which can be written more explicitly as

$$\xi\left[\alpha(X)\right] - \alpha\left([\xi, X]\right) = 0, \ \ \forall\, X \in \Gamma\left(T\mathscr{M}\right). \tag{4.1.2}$$

We sum up the preceding results in the Proposition:

Proposition 4.1.14. *A 1-form $\alpha \in \Omega^1(\mathscr{M})$ defined on the gravitational wave $\mathscr{M}$ admits a projection $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ on the Platonic screen $\bar{\mathscr{M}}$ if and only if the two following conditions are satisfied:*

1. *$\alpha \in \text{Ann}\ \xi$, i.e. $\alpha(\xi) = 0$,*
2. *$\mathcal{L}_\xi \alpha = 0$.*

The projected 1-form $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ is then defined as the 1-form satisfying the relation $\pi^*\bar{\alpha} = \alpha$. The 1-form α is said to be projectable on $\bar{\mathscr{M}}$.

Proposition 4.1.15. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $X \in \Gamma(T\mathscr{M})$ a vector field on $\mathscr{M}$. The 1-form dual to X, denoted $\alpha \equiv g(X) \in \Omega^1(\mathscr{M})$, is projectable if and only if:*

– *X is orthogonal to the wave vector field i.e. $g(X, \xi) = 0$*
– *X is ξ-invariant.*

Proof: The dual 1-form α must satisfy the conditions of Proposition 4.1.14:

1. $\alpha(\xi) = g(X, \xi) = 0$,
2. $\mathcal{L}_\xi \alpha = \mathcal{L}_\xi (g(X)) = (\mathcal{L}_\xi g)(X) + g([\xi, X]) = g([\xi, X]) = 0$

so that the vector field X must be orthogonal to ξ and ξ-invariant. Note that we used the Killing property of the wave vector field of Platonic waves. $\square$

Proposition 4.1.16. *Let $(\mathscr{M}, g, \xi)$ be a gravitational wave. The wave vector field ξ is affine geodesic if and only if $\mathcal{L}_\xi \psi = 0$. Equivalently, ξ is affine geodesic if and only if $d\psi(\xi) = 0$.*

Proof: Using the metric compatibility of the Levi-Civita connection ∇, one can write $\xi[g(\xi, X)] = g(\nabla_\xi \xi, X) + g(\xi, \nabla_\xi X)$ so that $\xi[\psi(X)] = g(\nabla_\xi \xi, X) + \psi(\nabla_\xi X)$. Reexpressing the second term on the right-hand side using the torsionfree condition leads to $g(\nabla_\xi \xi, X) = \xi[\psi(X)] - \psi(\nabla_X \xi) - \psi([\xi, X])$. Note that, according to the metric compatibility of ∇, one has $X[g(\xi, \xi)] = 2g(\nabla_X \xi, \xi) = 2\psi(\nabla_X \xi) = 0$, so that our expression becomes $g(\nabla_\xi \xi, X) = \xi[\psi(X)] - \psi([\xi, X]) \equiv (\mathcal{L}_\xi \psi)(X)$, $\forall X \in \Gamma(T\mathscr{M})$. The second equivalence is obtained using Cartan's formula $(\mathcal{L}_\xi \psi)(X) = d\psi(\xi, X) + X[\psi(\xi)] = d\psi(\xi, X)$. $\square$

Using Proposition 4.1.14, we can reformulate Lemma 2.1.6 as follows:

Proposition 4.1.17. *The wave covector field of a gravitational wave $\psi \equiv g(\xi)$ induces a well-defined absolute clock $\bar{\psi} \in \Omega^1(\bar{\mathscr{M}})$ on the Platonic screen defined by $\pi^* \bar{\psi} = \psi$.*

Proof: Condition 1 is immediate from the definition of ψ and the fact that ξ is null while condition 2 is ensured by Proposition 4.1.16. $\square$

Proposition 4.1.18. *The absolute clock $\bar{\psi}$ induces an involutive distribution on $\bar{\mathscr{M}}$.*

Proof: Let $X \in T\mathscr{M}$ be a vector field belonging to the kernel of ψ *i.e.* $\psi(X) = 0$. By definition of the absolute clock $\bar{\psi}$, we have $\pi^* \bar{\psi}(X) = \bar{\psi}(\pi_* X) = 0$, therefore $\pi_* X \in \text{Ker } \bar{\psi}$. The map $\pi : M \to \bar{\mathscr{M}}$ being a submersion, π_* is surjective so that any element $\bar{Y} \in T\bar{\mathscr{M}}$ can be written as $\bar{Y} = \pi_* Y$ with $Y \in T\mathscr{M}$. According to the precedent calculation, we see that any $\bar{X} \in \text{Ker } \bar{\psi}$ is the image of a vector field $X \in \text{Ker } \psi$, therefore the restriction of the pushforward $\pi_* : \text{Ker } \psi \to \text{Ker } \bar{\psi}$ is surjective.

Moreover, ψ defines an involutive distribution $\mathscr{M} \to \text{Ker } \psi$, so that for any couple of vector fields $X, Y \in \text{Ker } \psi$, we have $[X, Y] \in \text{Ker } \psi$. Therefore, $\psi([X, Y]) = \pi^* \bar{\psi}([X, Y]) = \bar{\psi}(\pi_* [X, Y]) = \bar{\psi}([\pi_* X, \pi_* Y]) = 0$, showing that $\bar{X}, \bar{Y} \in \text{Ker } \bar{\psi}$ implies $[\bar{X}, \bar{Y}] \in \text{Ker } \bar{\psi}$ and $\bar{\psi}$ then defines an involutive distribution on $\bar{\mathscr{M}}$. $\square$

Proposition 4.1.19. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave. The absolute clock $\bar{\psi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ induced by the wave covector field $\psi \in \Omega^1(\mathscr{M})$ is closed if and only if $(\mathscr{M}, g, \xi)$ is a Bargmann-Eisenhart wave.*

Proof: Starting from the definition of the absolute clock as $\psi \equiv \pi^*\bar{\psi}$ and acting on both sides with the exterior derivative gives $d\psi = d\left(\pi^*\bar{\psi}\right) = \pi^*\left(d\bar{\psi}\right)$, where we used that the exterior derivative commutes with the pullback. Now, imposing $d\bar{\psi} = 0$ leads to $d\psi = 0$ so the wave covector field must be closed. By definition of the exterior derivative of 1-form, the following equality stands for all vector fields $X, Y \in \Gamma(T\mathscr{M})$: $X[\psi(Y)] - Y[\psi(X)] - \psi([X,Y]) = 0$. Making the wave vector field appear and using the metric compatibility of the Levi-Civita connection leads to:

$$\begin{aligned} X[g(\xi, Y)] - Y[g(\xi, X)] - g(\xi, [X,Y]) &= 0 \\ g(\nabla_X \xi, Y) + g(\xi, \nabla_X Y) - g(\nabla_Y \xi, X) - g(\xi, \nabla_Y X) - g(\xi, [X,Y]) &= 0 \end{aligned}$$

Using the torsionfree condition allows to simplify the previous expression as: $g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = 0$. Furthermore, since $(\mathscr{M}, g, \xi)$ is a Platonic wave, the wave vector field is Killing and then satisfies $g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$, according to Proposition A.9.9. Putting these two equalities together leads to $g(\nabla_X \xi, Y) = 0$ $\forall X, Y \in \Gamma(T\mathscr{M})$, so that ξ is parallel with respect to ∇ and $(\mathscr{M}, g, \xi)$ is thus a Bargmann-Eisenhart wave. $\square$

4.2 Aristotelian structure embedded in a Kundt wave

According to Proposition 4.1.18, the most general Leibnizian structure that can be induced from the metric structure of a gravitational wave is an Aristotelian structure, since the absolute clock $\bar{\psi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ obtained by projecting the wave covector field $\psi \in \Omega^1(\mathscr{M})$ necessarily satisfies the Frobenius Criterion. In this Section, we show that Kundt waves are the most general class of gravitational waves allowing such a projection.

Notation 4.2.1. We retain the terminology of Section 3.2 and refer to a vector field $\bar{X} \in \text{Ker}\, \bar{\psi}$ as a spacelike vector field. On the other hand, a vector field $X \in \text{Ker}\, \psi$, thus satisfying $\psi(X) = g(\xi, X) = 0$, will be referred to as ξ-orthogonal. Note that any lift of a spacelike vector field is ξ-orthogonal while any projectable ξ-orthogonal vector field projects onto a spacelike vector field on the Platonic screen.

Proposition 4.1.14 set out the necessary and sufficient conditions a 1-form on a gravitational wave must satisfy in order to admit a well-defined projection on the Platonic screen. These conditions generalise straightforwardly to a covariant bilinear form, so that the following Proposition holds

Proposition 4.2.2. *A covariant bilinear form* $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ *defined on the gravitational wave* $\mathscr{M}$ *admits a projection* $\bar{g} \in \Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$ *on the Platonic screen* $\bar{\mathscr{M}}$ *if and only if the two following conditions are satisfied:*

1. $\xi \in \mathrm{Rad}\ g$*, i.e.* $g(\xi) = 0$*,*
2. $\mathcal{L}_\xi g = 0$.

If such conditions are met, the projection $\bar{g}$ is defined as $\pi^*\bar{g} = g$, where π^* denotes the generalised pullback (*cf.* Definition A.2.7). However, a dramatic consequence of Proposition 4.2.2 is that it prevents the possibility to define a covariant metric on the Platonic screen by projecting a (pseudo)-Riemannian metric, since only degenerate covariant metrics are projectable. In order to circumvent this drawback, we are led to define a degenerate relativistic metric [3] as follows:

Definition 4.2.3 (Relativistic spatial metric)**.** *Let* $(\mathscr{M}, g, \xi)$ *be a gravitational wave. The relativistic spatial metric* $\gamma \in \Gamma\left(\vee^2 \mathrm{Ker}\ \psi\right)$ *is defined as the restriction of the covariant metric* $g \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ *to* ξ*-orthogonal vector fields i.e.*

$$\gamma(X, Y) \equiv g(X, Y) \tag{4.2.3}$$

where $X, Y \in \mathrm{Ker}\ \psi$*.*

Acting on ξ-orthogonal vector fields, the metric γ endows the wavefront worldvolumes (*i.e.* the integral submanifolds of the distribution induced by the kernel of ψ) with a notion of distance. It should be noted that γ is degenerate, as can be seen from $\gamma(\xi, X) = g(\xi, X) = \psi(X)\, 0$, where we used that both ξ and X are ξ-orthogonal, so that $\xi \in \mathrm{Rad}\ \gamma$. In fact, since $\mathrm{Rad}\ g = \emptyset$, we have $\mathrm{Rad}\ \gamma = \mathrm{Span}\ \xi$. [4]

Proposition 4.2.4. *Let* $(\mathscr{M}, g, \xi)$ *be a gravitational wave. The relativistic spatial metric* $\gamma \in \Gamma\left(\vee^2 \mathrm{Ker}\ \psi\right)$ *admits a projection on the absolute spaces of the Platonic screen* $\bar{\mathscr{M}}$ *if and only if* $(\mathscr{M}, g, \xi)$ *is a Kundt wave. The metric* γ *then projects onto a positive-definite metric* $\bar{\gamma} \in \Gamma\left(\vee^2 \mathrm{Ker}\ \bar{\psi}\right)$ *.*

3. Strictly speaking, γ is not a metric since γ_x at a point $x \in \mathscr{M}$ is not defined on the whole of $T_x\mathscr{M}$. We will however keep this denomination and once again understand the term "metric" in a broader sense than the usual.

4. In the language of [77], γ is thus a Carrollian metric.

Proof: According to Proposition 4.2.2, the relativistic spatial metric $\gamma \in \Gamma\left(\vee^2 \operatorname{Ker} \psi\right)$ is projectable if and only if the two following properties are satisfied:

1. $\xi \in \operatorname{Rad} \gamma$,
2. $\mathcal{L}_\xi \gamma = 0$.

The first property has already been shown to hold for a generic gravitational wave. As for the second, it is equivalent to $\mathcal{L}_\xi g\,(X, Y) = 0$ for all $X, Y \in \operatorname{Ker} \psi$, which will be shown below to hold if and only if $(\mathscr{M}, g, \xi)$ is a Kundt wave (*cf.* Proposition 2 of Lemma 4.4.12). Now, assuming that $(\mathscr{M}, g, \xi)$ is a Kundt wave, γ is projectable and there exists a unique covariant metric $\bar{\gamma} \in \Gamma\left(\vee^2 \operatorname{Ker} \bar{\psi}\right)$ such that $\pi^* \bar{\gamma} = \gamma$, where π^* stands for the generalised pullback (*cf.* Definition A.2.7). Explicitly, the action of $\bar{\gamma}$ on two spacelike vector fields $\bar{X}, \bar{Y} \in \operatorname{Ker} \bar{\psi}$ is given by:

$$\bar{\gamma}\left(\bar{X}, \bar{Y}\right) \circ \pi = \gamma\,(X, Y) \tag{4.2.4}$$

where X and $Y \in \operatorname{Ker} \psi$ are arbitrary lifts of $\bar{X}$ and $\bar{Y}$ respectively. Note that X and Y are necessarily ξ-orthogonal since $\bar{X}$ and $\bar{Y}$ are assumed to be spacelike. Furthermore, the property $\xi \in \operatorname{Rad} \gamma$ ensures that $\bar{\gamma}\left(\bar{X}, \bar{Y}\right)$ is independent on the choice of lifts X and Y. Finally, the fact that γ is positive semi-definite ensures the positive definiteness of $\bar{\gamma}$. $\square$

Comparing the results obtained with Definitions 3.2.1 and 3.2.15 of Lebnizian and Aristotelian structures, respectively, we can sum up Propositions 4.1.17, 4.1.18 and 4.2.4 as:

Proposition 4.2.5. *A gravitational wave induces an Aristotelian structure on its Platonic screen if and only if it is a Kundt wave.*

Focusing on the class of Platonic waves, the embedding of nonrelativistic metric structures can be summarised in the following Table (*cf.* Propositions 4.1.17, 4.1.18, 4.1.19 and 4.2.5):

Gravitational wave	Induced metric structure
Platonic wave $(\mathscr{M}, g, \xi)$	Aristotelian structure $\left(\bar{\mathscr{M}}, \bar{\psi}, \bar{h}\right)$
Bargmann-Eisenhart wave $(\mathscr{M}, g, \xi)$	Augustinian structure $\left(\bar{\mathscr{M}}, \bar{\psi}, \bar{h}\right)$

Table 4.1: Embedding of nonrelativistic metric structures inside Platonic waves

Now, the nonrelativistic spatial metric $\bar{\gamma} \in \Gamma\left(\vee^2 \operatorname{Ker} \bar{\psi}\right)$ can be used in order to define a degenerate contravariant metric $\bar{h} \in \Gamma\left(\vee^2 T\bar{\mathscr{M}}\right)$ (*cf.* Proposition 3.2.3). However, one can wonder if the nonrelativistic contravariant metric $\bar{h}$ cannot be obtained directly from the projection of a relativistic contravariant metric on $\mathscr{M}$. To answer this question, we will need the following Proposition which straightforwardly extends the projectability condition of a vector field (*cf.* Proposition 4.1.8):

Proposition 4.2.6. *A contravariant bilinear form $h \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ defined on the gravitational wave $\mathscr{M}$ admits a projection $\bar{h} \in \Gamma\left(\vee^2 T\bar{\mathscr{M}}\right)$ on the Platonic screen $\bar{\mathscr{M}}$ if and only if there exists a vector field $X \in \Gamma\left(T\mathscr{M}\right)$ such that*

$$\mathcal{L}_\xi h = X \vee \xi. \tag{4.2.5}$$

This condition can be equivalently expressed as

$$\left(\mathcal{L}_\xi h\right)(\alpha, \beta) = 0,\ \forall \alpha, \beta \in \operatorname{Ann} \xi. \tag{4.2.6}$$

We conclude this Section by showing that the inverse metric $g^{-1} \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ of a gravitational wave is projectable if and only if $(\mathscr{M}, g, \xi)$ is a Kundt wave. The proof will make use of the Brinkmann coordinates introduced in Section 2.1.2. Let α and $\beta \in \operatorname{Ann} \xi$ be two 1-forms on $\mathscr{M}$ annihilating ξ. In Brinkmann coordinates, this condition reads $\alpha_u = \beta_u = 0$, so that condition (4.2.6) reads $\mathcal{L}_\xi g^{-1}(\alpha, \beta) = \partial_u g^{tt}\, \alpha_t\, \beta_t + \partial_u g^{ti}\, \alpha_t\, \beta_i + \partial_u g^{ij}\, \alpha_i\, \beta_j = 0$. The two first terms vanish for a generic gravitational wave (*cf.* eq. (2.1.2)), so that, in Brinkmann coordinates, the consistency condition reads $\partial_u g^{ij} = 0$. This condition singles out the class of Kundt waves (*cf.* Proposition 2.2.16) so that the inverse metric g^{-1} of a gravitational wave projects well onto the Platonic screen if and only if it is a Kundt wave. It can furthermore be shown that the metric $\bar{h}$ defined is degenerate and that $\operatorname{Rad} \bar{h} = \operatorname{Span} \bar{\psi}$, by computing $\bar{h}\left(\bar{\psi}, \bar{\alpha}\right) = g^{-1}\left(\pi^*\bar{\psi}, \pi^*\bar{\alpha}\right) = g^{-1}\left(\psi, \pi^*\bar{\alpha}\right) = \pi^*\bar{\alpha}\left(\xi\right) = 0$ where we used the definition of ψ as $g\left(\xi\right)$ and the fact that the pullback π^* maps 1-forms of $\bar{\mathscr{M}}$ into elements of $\operatorname{Ann} \xi$. Therefore, $\bar{h}\left(\bar{\psi}, \bar{\alpha}\right)$ vanishes $\forall\, \bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$, which implies $\bar{\psi} \in \operatorname{Rad} \bar{h}$. Since $\operatorname{Ker}\ (\pi^*) = \operatorname{Rad} g^{-1} = \emptyset$, one gets $\operatorname{Span} \bar{\psi} = \operatorname{Rad} \bar{h}$.

4.3 Ehresmann connections on a gravitational wave

The previous Section has described how the (nondegenerate) metric structure of a Kundt wave projects onto a (degenerate) Aristotelian structure on the Platonic screen. The next logical step is then to investigate how a notion of parallelism (provided by the Levi-Civita connection) on a gravitational wave can be lowered downto the Platonic screen. As we will see, such an endeavour can only be achieved for a restricted class of gravitational waves, namely Platonic waves. This constitutes the subject of the next Section. In the meantime, we discuss a few additional notions revolving around the concept of relativistic field of observers, aiming to draw connections with the nonrelativistic avatars of these concepts, whose importance in nonrelativistic physics has been emphasised in Chapter 3. In the process, we introduce Bargmann bases (*cf.* Definition 4.3.9) which will be encountered again in Section 6.3 and also investigate the link between observers and Ehresmann connections on the principal bundle (4.1.1).

Definition 4.3.1 ((Light-like) Relativistic field of observers)**.** *A relativistic field of observers is a vector field $N \in \Gamma(T\mathscr{M})$ such that $\psi(N) = 1$. The space of all relativistic fields of observers on $\mathscr{M}$ is denoted $FO(\mathscr{M})$. We denote $\overset{N}{A}$ the 1-form dual to N via the metric g. If N satisfies the additional condition that $\overset{N}{A}(N) \equiv g(N,N) = 0$, then N is called a relativistic field of light-like observers.*

Proposition 4.3.2. *Let $N \in FO(\mathscr{M})$ be a projectable relativistic field of observers. Then its projection $\bar{N} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ on the Platonic screen is a field of observers.*

Proof: The proof is obtained straightforwardly by recalling that the wave covector field ψ projects onto the absolute clock $\bar{\psi}$ so that $\pi^*\bar{\psi} = \psi$. Inserting this relation inside the condition $\psi(N) = 1$ leads to $\left(\pi^*\bar{\psi}\right)(N) = \bar{\psi}(\pi_* N) \circ \pi = \bar{\psi}\left(\bar{N}\right) \circ \pi = 1$. □

Definition 4.3.3 (Spacelike projection of vector fields)**.** *Let $N \in FO(\mathscr{M})$ be a relativistic field of light-like observers. The field of endomorphisms $P^N : \Gamma(T\mathscr{M}) \to \mathrm{Ker}\ \psi \cap \mathrm{Ker}\ \overset{N}{A}$ defined as*

$$P^N(X) = X - \psi(X)\,N - \overset{N}{A}(X)\,\xi \tag{4.3.7}$$

where X is any vector field, is called the spacelike projector of vector fields along N.

Note that $\mathrm{Ker}\ P^N = \mathrm{Span}\{\xi, N\}$.

The transpose of P^N, denoted $\bar{P}^N : \Gamma(T\mathscr{M}) \to \mathrm{Ann}\ N \cap \mathrm{Ann}\ \xi$, can also be defined as $\bar{P}^N(\alpha)(X) = \alpha\left(P^N(X)\right)$, where $\alpha \in \Omega^1(\mathscr{M})$ and $X \in \Gamma(T\mathscr{M})$. The explicit form of $\bar{P}^N$ is then given by

$$\bar{P}^N(\alpha) = \alpha - \alpha(N)\,\psi - \alpha(\xi)\,\overset{N}{A} \tag{4.3.8}$$

and $\mathrm{Ker}\ \bar{P}^N = \mathrm{Span}\left\{\psi, \overset{N}{A}\right\}$.

Lemma 4.3.4. *Let $N \in FO(\mathscr{M})$ be a projectable relativistic field of light-like observers and $X \in \Gamma(T\mathscr{M})$ a projectable vector field. The following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{P^N} & P^N(X) \\ \pi_* \downarrow & & \downarrow \pi_* \\ \bar{X} & \xrightarrow{P^{\bar{N}}} & P^{\bar{N}}\left(\bar{X}\right) \end{array}$$

with

- $\bar{N} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ *the field of observers obtained by projection of N on the Platonic screen*
- $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ *the projection of X on the Platonic screen*
- $P^N : \Gamma(T\mathscr{M}) \to \mathrm{Ker}\ \psi \cap \mathrm{Ker}\ \overset{N}{A}$ *the field of endomorphisms defined by eq.*(4.3.7) *associated to N*
- $P^{\bar{N}} : \Gamma\left(T\bar{\mathscr{M}}\right) \to \mathrm{Ker}\ \bar{\psi}$ *the field of endomorphisms defined by eq.*(3.2.7) *associated to $\bar{N}$.*

Proof: The proof is straightforwardly derived by projecting (4.3.7)

$$\begin{aligned} \pi_*\left(P^N(X)\right) &= \pi_*\left(X - \psi(X)\, N - \overset{N}{A}(X)\,\xi\right) \\ &= \bar{X} - \bar{\psi}\left(\bar{X}\right)\bar{N} \\ &= P^{\bar{N}}\left(\bar{X}\right) \end{aligned}$$

where Lemma 4.1.13 has been used. □

Note that a similar statement can be made regarding the transpose $\bar{P}^N$ whose projection coincides with the transpose $\bar{P}^{\bar{N}}$.

Definition 4.3.5 (Relativistic covariant transverse metric)**.** *Let $(\mathscr{M}, g, \xi)$ be a gravitational wave and $N \in FO(\mathscr{M})$ a relativistic field of light-like observers on $\mathscr{M}$. The transverse metric $\overset{N}{\gamma} \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ is defined by its action on vector fields $X, Y \in \Gamma(TM)$ as*

$$\overset{N}{\gamma}(X, Y) = g\left(P^N(X), P^N(Y)\right) \tag{4.3.9}$$

where P^N stands for the spacelike projector associated to N (cf. Definition 4.3.3).

Explicitly, the relativistic transverse metric reads $\overset{N}{\gamma} \equiv g - \psi \otimes \overset{N}{A} - \overset{N}{A} \otimes \psi$.

Proposition 4.3.6. *Let $(\mathscr{M}, g, \xi)$ be a Kundt wave. The relativistic transverse metric $\overset{N}{\gamma} \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ associated to the projectable relativistic field of observers $N \in FO(\mathscr{M})$*

is projectable. Its projection is the transverse metric $\overset{\bar{N}}{\gamma} \in \Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$ *associated to the projection* $\bar{N} \in FO\left(\bar{\mathscr{M}}\right)$ *of* N*.*

Proof: In order to be projectable, the relativistic transverse metric $\overset{N}{\gamma}$ must satisfy the two following properties (*cf.* Proposition 4.2.2):

$$\begin{cases} \mathcal{L}_\xi \overset{N}{\gamma} = 0 \\ \overset{N}{\gamma}(\xi, X) = 0, \ \forall\, X \in \Gamma(T\mathscr{M}) \end{cases} \tag{4.3.10}$$

The second condition is straightforward from the fact that $\xi \in \text{Ker } P^N$ while the first property can be seen to follow from $\mathcal{L}_\xi g(V, W) = 0$ for all vector fields $V, W \in \text{Ker } \psi$ (*cf.* Proposition 2 of Lemma 4.4.12 below). In particular, since $P^N(Z) \in \text{Ker } \psi$, $\forall\, Z \in \Gamma(T\mathscr{M})$, one can choose $V = P^N(X)$ and $W = P^N(Y)$, so that:

$$\xi\left[g\left(P^N(X), P^N(Y)\right)\right] = g\left(\left[\xi, P^N(X)\right], P^N(Y)\right) + g\left(P^N(X), \left[\xi, P^N(Y)\right]\right).$$

Now, expanding $P^N([\xi, X])$ as

$$\begin{aligned} P^N([\xi, X]) &= [\xi, X] - \psi([\xi, X])\, N - \overset{N}{A}([\xi, X])\,\xi \\ &= [\xi, X] - \xi[\psi(X)]\, N + d\psi(\xi, X)\, N - \xi\left[\overset{N}{A}(X)\right]\xi + d\overset{N}{A}(\xi, X)\,\xi \\ &= [\xi, X] - [\xi, \psi(X)\, N] + \psi(X)[\xi, N] - \left[\xi, \overset{N}{A}(X)\,\xi\right] + \overset{N}{A}(X)[\xi, \xi] + \overset{N}{F}(\xi, X)\,\xi \\ &= \left[\xi, P^N(X)\right] + f\psi(X)\,\xi + \overset{N}{F}(\xi, X)\,\xi \end{aligned}$$

where in the first step we used the definition of the exterior derivative of a 1-form; in the second step, the term $d\psi(\xi, X)\, N$ is seen to vanish by application of Proposition 4.1.16 (ξ being assumed to be affine geodesic) and we introduced the 2-form $\overset{N}{F} \equiv d\overset{N}{A} \in \Omega^2(\mathscr{M})$; in the third step, one made use of the projectability of N, so that there exists a function $f \in C^\infty(\mathscr{M})$ satisfying $[\xi, N] = f\xi$.
Since $P^N([\xi, X])$ and $\left[\xi, P^N(X)\right]$ only differs by terms along the wave vector field ξ, we get $g\left(\left[\xi, P^N(X)\right], P^N(Y)\right) = g\left(P^N([\xi, X]), P^N(Y)\right)$, since $P^N(Y) \in \text{Ker } \psi$. Repeating the operation with X and Y in reverse order allows to write

$$\xi\left[\overset{N}{\gamma}(X, Y)\right] = \overset{N}{\gamma}([\xi, X], Y) + \overset{N}{\gamma}(X, [\xi, Y]), \ \textit{i.e.}\ \ \mathcal{L}_\xi \overset{N}{\gamma} = 0.$$

Now, since γ is projectable, there exists a unique covariant metric $\overset{\bar{N}}{\gamma} \in \Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$ such that $\pi^*\overset{\bar{N}}{\gamma} = \overset{N}{\gamma}$, where π^* stands for the generalised pullback (*cf.* Definition

A.2.7). Explicitly, the action of $\overset{\bar{N}}{\gamma}$ on two vector fields $\bar{X}, \bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ is given by:

$$\overset{\bar{N}}{\gamma}\left(\bar{X}, \bar{Y}\right) \circ \pi = \overset{N}{\gamma}\left(X, Y\right)$$

where X and $Y \in \Gamma\left(T\mathscr{M}\right)$ are arbitrary lifts of $\bar{X}$ and $\bar{Y}$ respectively. Note that $\overset{\bar{N}}{\gamma}\left(\bar{X}, \bar{Y}\right)$ is independent on the choice of lifts X and Y, since $\xi \in \operatorname{Rad} \overset{N}{\gamma}$.
Finally, one needs to make contact with Definition 3.2.11 for a nonrelativistic transverse metric on $\bar{\mathscr{M}}$. This is done by remarking that $\overset{N}{\gamma}\left(X, Y\right) = \gamma\left(P^N\left(X\right), P^N\left(Y\right)\right)$, with $\gamma \in \Gamma\left(\vee^2 \operatorname{Ker} \psi\right)$ defined by eq.(4.2.3). Explicitly,

$$\begin{aligned} \overset{\bar{N}}{\gamma}\left(\bar{X}, \bar{Y}\right) \circ \pi &= \gamma\left(P^N\left(X\right), P^N\left(Y\right)\right) \\ &= \bar{\gamma}\left(\pi_*\left(P^N\left(X\right)\right), \pi_*\left(P^N\left(Y\right)\right)\right) \circ \pi \\ &= \bar{\gamma}\left(P^{\bar{N}}\left(\bar{X}\right), P^{\bar{N}}\left(\bar{Y}\right)\right) \circ \pi \end{aligned}$$

where Lemma 4.3.4 has been used along with eq.(4.2.4). The last line thus reproduces eq.(4.3.9) so that $\overset{\bar{N}}{\gamma}$ is the nonrelativistic transverse metric associated to the field of observers $\bar{N}$. $\square$

Definition 4.3.7 (Relativistic contravariant transverse metric)**.** *Let $(\mathscr{M}, g, \xi)$ be a gravitational wave and $N \in FO\left(\mathscr{M}\right)$ a relativistic field of light-like observers on $\mathscr{M}$. The contravariant transverse metric $\overset{N}{h} \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ is defined by its action on 1-forms $\alpha, \beta \in \Gamma\left(T^*M\right)$ as*

$$\overset{N}{h}\left(\alpha, \beta\right) = g^{-1}\left(\bar{P}^N\left(\alpha\right), \bar{P}^N\left(\beta\right)\right) \tag{4.3.11}$$

where $\bar{P}^N$ stands for the transpose of the spacelike projector associated to N.

Explicitly, the form of $\overset{N}{h}$ is given by $\overset{N}{h} \equiv g^{-1} - N \otimes \xi - \xi \otimes N$.

Proposition 4.3.8. *Let $(\mathscr{M}, g, \xi)$ be a Kundt wave. The relativistic contravariant transverse metric $\overset{N}{h} \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ associated to the relativistic field of observers $N \in FO\left(\mathscr{M}\right)$ is projectable, its projection being the contravariant metric $\bar{h} \in \Gamma\left(\vee^2 T\bar{\mathscr{M}}\right)$.*

Proof: In order for $\overset{N}{h}$ to be projectable, there must exist a vector field $X \in \Gamma\left(T\mathscr{M}\right)$ such that $\mathcal{L}_\xi \overset{N}{h} = X \vee \xi$ (*cf.* Proposition 4.2.6). Using the explicit expression of $\overset{N}{h}$, one obtains $\mathcal{L}_\xi \overset{N}{h} = \mathcal{L}_\xi g^{-1} - \mathcal{L}_\xi \xi \vee N - \xi \vee \mathcal{L}_\xi N$. Since g^{-1} is projectable, there exists a vector field $Y \in \Gamma\left(T\mathscr{M}\right)$ such that $\mathcal{L}_\xi g^{-1} = Y \vee \xi$, so that $\mathcal{L}_\xi \overset{N}{h} = \left(Y - \mathcal{L}_\xi N\right) \vee \xi$, and $\overset{N}{h}$ is therefore projectable. Furthermore, since g^{-1} and $\overset{N}{h}$ differ only by terms of the form $X \vee \xi$, their projections coincide, *i.e.* $\pi_* \overset{N}{h} = \pi_* g^{-1} = \bar{h}$. $\square$

Definition 4.3.9 (Bargmann basis). *Let $(\mathscr{M}, g, \xi)$ be a gravitational wave. A Bargmann basis of the tangent space $T_x\mathscr{M}$ at a point $x \in \mathscr{M}$ is an ordered basis $B_x = \{\xi_x, N_x, e_{1x}, \ldots, e_{dx}\}$ with ξ_x the wave vector at the point $x \in \mathscr{M}$, N_x the tangent vector of a light-like relativistic observer and $\{e_{1x}, \ldots, e_{dx}\}$ a basis of* $\mathrm{Ker}\ \psi_x \cap \mathrm{Ker}\ \overset{N}{A}_x$ *(where $\overset{N}{A}_x \equiv g_x(N_x)$ is the 1-form dual to N_x) which is orthonormal with respect to g_x.*

Explicitly, the basis $B_x = \{\xi_x, N_x, e_{1x}, \ldots, e_{dx}\}$ must satisfy the conditions:

1. $g_x(\xi_x, N_x) = 1$
2. $g_x(\xi_x, e_{ix}) = 0\,, \forall\, i \in \{1, \ldots, d\}$
3. $g_x(N_x, N_x) = 0$
4. $g_x(N_x, e_{ix}) = 0\,, \forall\, i \in \{1, \ldots, d\}$
5. $g_x(e_{ix}, e_{jx}) = \delta_{ij}\,, \forall\, i, j \in 1, \ldots, d.$

A basis of $T^*_x\mathscr{M}$ dual to $B_x = \{\xi_x, N_x, e_{ix}\}$ is given by $B^*_x \equiv \left\{\overset{N}{A}_x, \psi_x, \theta^i_x\right\}$, where the d one-forms θ^i_x satisfy the requirement: $\theta^i_x(e_{jx}) = \delta^i_j$.

The denomination Bargmann basis in Definition 4.3.9 is justified by the following Proposition :

Proposition 4.3.10 (*cf.* [24]). *At each point $x \in \mathscr{M}$, the set of endomorphisms of $T_x\mathscr{M}$ mapping each Bargmann basis into another one forms a group isomorphic to the homogeneous Galilei group* Gal_0 *in the $d+2$-dimensional faithful representation inherited from that of the Bargmann group*[5].

Proof: Let us denote by $T : T_x\mathscr{M} \to T_x\mathscr{M}$ one of the endomorphisms considered. Since T maps bases into bases, it must be a vector space isomorphism so that it can be represented by an element of $GL(T_x\mathscr{M})$. Taking into account that ξ_x is preserved by T, we have the following invertible matrix

$$T \equiv \begin{pmatrix} 1 & 0 & 0 \\ d & a & \mathbf{b} \\ \mathbf{f} & \mathbf{c} & \mathbf{R} \end{pmatrix} \qquad (4.3.12)$$

5. See Section II.A of [24] for more details.

where a, $d \in \mathbb{R}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{c} \in \mathbb{R}^d$ and $\mathbf{R} \in GL\left(\mathbb{R}^d\right)$. Let $B_x = \{\xi_x, N_x, e_{ix}\}$ be a Bargmann basis of $T_x\mathscr{M}$, the basis $T(B_x) = \{\xi'_x, N'_x, e'_{ix}\}$ reads (dropping the index x for notational simplicity):

$$T\begin{pmatrix}\xi\\N\\e_i\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ d & a & \mathbf{b}\\ \mathbf{f} & \mathbf{c} & \mathbf{R}\end{pmatrix}\begin{pmatrix}\xi\\N\\e_i\end{pmatrix} = \begin{pmatrix}\xi\\ d\xi + aN + \mathbf{b}^i e_i\\ \mathbf{f}_j\xi + \mathbf{c}_j N + \mathbf{R}^i_{\ j} e_i\end{pmatrix}. \tag{4.3.13}$$

Requiring that $T(B_x)$ is a Bargmann basis (Conditions 1-5 following Definition 4.3.9) imposes that T satisfy:

1. $g_x(\xi_x, N_x) = 1 \Rightarrow a = 1$
2. $g_x(\xi_x, e_{ix}) = 0\,, \forall\, i \in \{1, \ldots, d\} \Rightarrow \mathbf{c}_i = 0$
3. $g_x(N_x, N_x) = 0 \Rightarrow d = -\frac{1}{2}\mathbf{b}^T\mathbf{b}$
4. $g_x(N_x, e_{ix}) = 0\,, \forall\, i \in \{1, \ldots, d\} \Rightarrow \mathbf{f} = -\mathbf{b}^T\mathbf{R}$
5. $g_x(e_{ix}, e_{jx}) = \delta_{ij}\,, \forall\, i, j \in \{1, \ldots, d\} \Rightarrow \mathbf{R} \in O(d)$.

The set of matrices representing the set of isomorphisms T is then of the form

$$T = \begin{pmatrix}1 & 0 & 0\\ -\frac{1}{2}\mathbf{b}^T\mathbf{b} & 1 & \mathbf{b}\\ -\mathbf{b}^T\mathbf{R} & 0 & \mathbf{R}\end{pmatrix} \tag{4.3.14}$$

with $\mathbf{b} \in \mathbb{R}^d$ and $\mathbf{R} \in O(d)$. This set of matrices form a subgroup of $GL\left(\mathbb{R}^{1,d+1}\right)$ isomorphic to the homogeneous Galilei group Gal_0. The homogeneous Galilei group therefore acts regularly on the space of Bargmann basis via the group action:

$$\{\xi, N, e_i\} \overset{h}{\mapsto} \left\{\xi, N + \mathbf{b}^i e_i - \frac{1}{2}\mathbf{b}^T_i\mathbf{b}^i\xi, \mathbf{R}^j_{\ i} e_j - \mathbf{b}^T_j\mathbf{R}^j_{\ i}\xi\right\}. \tag{4.3.15}$$

□

Both Definition 4.3.9 and Proposition 4.3.10 can be generalised in a straightforward way from the tangent space at a point of $\mathscr{M}$ to the tangent bundle of $\mathscr{M}$. A Bargmann basis of $T\mathscr{M}$ is then defined as the ordered set of fields $B = \{\xi, N, e_1, \ldots, e_n\}$ with N a field of observers and $\{e_1, \ldots, e_n\}$ a basis of $\mathrm{Ker}\ \psi \cap \mathrm{Ker}\ \overset{N}{A}$ (where $\overset{N}{A} \equiv g(N)$ is the 1-form dual to N), orthonormal with respect to the metric g. Two Bargmann bases $\{\xi', N', e'_i\}$ and $\{\xi, N, e_i\}$ are mapped via a local transformation where $\mathbf{R} : \mathscr{M} \to O(d)$ now parameterise a local spatial rotation and $\mathbf{b}^i : \mathscr{M} \to \mathbb{R}^d$ a local null rotation. Explicitly, one has:

$$\begin{cases} N' = N + \mathbf{b}^i e_i - \frac{1}{2}\mathbf{b}_i^T \mathbf{b}^i \xi \\ e'_i = \mathbf{R}^j{}_i e_j - \mathbf{b}_j^T \mathbf{R}^j{}_i \xi \end{cases}$$

where the first expression is a relativistic Milne boost (*cf.* Proposition 4.3.14 below).

The appearance of the nonrelativistic homogeneous Galilei group can seem peculiar, given that we work in a purely relativistic context. To reformulate the previous result, Proposition 4.3.10 showed that the group permuting the light-cone bases of a Lorentzian spacetime while preserving one of the light-like direction is the homogeneous Galilei group, in its Bargmann representation. The origin of this fact can be traced back to the "light-like" embedding of the Bargmann group inside the Poincaré group (*cf.* [140, 55, 141, 142, 57]).

Bargmann bases can be used in order to reinterpret some of the notions previously introduced. As such, if we let $B = \{\xi, N, e_i\}$ be a Bargmann with dual basis $B^* = \left\{\overset{N}{A}, \psi, \theta^i\right\}$, the following relations hold

$$\begin{cases} P^N(X) = \theta^i(X)\, e_i \\ \bar{P}^N(\alpha) = \alpha(e_i)\, \theta^i \\ \overset{N}{\gamma} = \theta^i \vee \theta^j\, \delta_{ij} \\ \overset{N}{h} = e_i \vee e_j\, \delta^{ij} \end{cases} \tag{4.3.16}$$

with $X \in \Gamma(T\mathscr{M})$ and $\alpha \in \Omega^1(\mathscr{M})$.

Proposition 4.3.11. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $N \in FO(\mathscr{M})$ be a ξ-invariant relativistic field of observers. Then the dual 1-form $\overset{N}{A} \equiv g(N) \in \Omega^1(\mathscr{M})$ is an Ehresmann connection for the fiber bundle* $\mathbb{R} \hookrightarrow \mathscr{M} \xrightarrow{\pi} \bar{\mathscr{M}}$.

Proof: According to the Definition A.7.1, the dual 1-form $\overset{N}{A}$ must satisfy the two following criteria:

1. $\overset{N}{A}(\xi) = 1$ (Action on vertical vectors)
2. $\mathcal{L}_\xi \overset{N}{A} = 0$ (Equivariance).

Criterion 1 directly follows from the fact that $N \in FO(\mathscr{M})$ so that $\psi(N) = \overset{N}{A}(\xi) = 1$ (note than 1 is the generator of the Abelian Lie algebra $\mathbb{R}$). As for criterion 2, one starts from the fact $\mathcal{L}_\xi g = 0$, so that $\xi[g(Y, Z)] = g([\xi, Y], Z) + g(Y, [\xi, Z])$, $\forall Y, Z \in \Gamma(T\mathscr{M})$. In particular, $\xi[g(N, Z)] = g([\xi, N], Z) + g(N, [\xi, Z])$, $\forall Z \in \Gamma(T\mathscr{M})$. The second term on the right-hand side vanishes according to the hypothesis $\mathcal{L}_\xi N = 0$ so that the equality becomes $\xi\left[\overset{N}{A}(Z)\right] = \overset{N}{A}([\xi, Z])\ \forall Z \in \Gamma(T\mathscr{M})$ *i.e.* $\mathcal{L}_\xi \overset{N}{A} = 0$. $\square$

Proposition 4.3.11 shows how relativistic fields of observers preserved by the null direction ξ induce a notion of horizontality on the fiber bundle $\mathbb{R} \hookrightarrow \mathscr{M} \xrightarrow{\pi} \bar{\mathscr{M}}$. The subbundle of horizontal vector fields with respect to the connection $\overset{N}{A}$ will be denoted as $\overset{N}{H}(T\mathscr{M})$. Light-like field of observers preserved by the null direction ξ can be seen as horizontal vector fields with respect to their dual Ehresmann connection. This further justifies the use of light-like fields of observers on $\mathscr{M}$ as lifts of fields of observers on the Platonic screen.

Definition 4.3.12 (Horizontal projection)**.** *Let $X \in \Gamma(T\mathscr{M})$ be a vector field on $\mathscr{M}$. The field of endomorphisms $\tilde{P}^N : \Gamma(T\mathscr{M}) \to \overset{N}{H}(T\mathscr{M})$ defined as $\tilde{P}^N(X) = X - \overset{N}{A}(X)\,\xi$, with $X \in \Gamma(T\mathscr{M})$, is called horizontal projection of vector fields with respect to the connection $\overset{N}{A}$.*

Proposition 4.3.13. *Let $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ be a vector field on $\bar{\mathscr{M}}$ and $X \in \Gamma(T\mathscr{M})$ the horizontal lift of $\bar{X}$ with respect to the Ehresmann connection $\overset{N}{A} \in \Omega^1(\mathscr{M})$. Then X is a ξ-invariant vector field on $\mathscr{M}$.*

Proof: Being a lift, the vector field X is necessarily projectable, *i.e.* there exists a function $\lambda \in C^\infty(\mathscr{M})$ such that $[\xi, X] = \lambda\xi$. Using the equivariance of $\overset{N}{A}$, one writes $\xi\left[\overset{N}{A}(X)\right] = \overset{N}{A}([\xi, X])$. The left-hand side vanishes since X is horizontal while the right-hand side reads $\overset{N}{A}([\xi, X]) = \overset{N}{A}(\lambda\xi) = \lambda\overset{N}{A}(\xi) = \lambda = 0$, hence X is ξ-invariant. $\square$

Proposition 4.3.14 (Relativistic Milne boost)**.** *Let N and N' be two relativistic fields of light-like observers. There exists a 1-form $\chi \in \Omega^1(\mathscr{M})$ such that*

$$N' = N + \overset{N}{h}(\chi) - \frac{1}{2}\overset{N}{h}(\chi, \chi)\,\xi. \tag{4.3.17}$$

The relativistic fields of light-like observers N and N' are said to be related by a relativistic Milne boost parameterised by the 1-form χ.

The choice of terminology although non-standard, can be justified by the obvious following fact:

Proposition 4.3.15. *Let $(\mathscr{M}, g, \xi)$ be a gravitational wave and $N', N \in FO(\mathscr{M})$ be two relativistic fields of light-like observers related by a relativistic Milne boost parameterised by* *the projectable 1-form $\chi \in \Omega^1(\mathscr{M})$. Then the vector fields $\bar{N}'$ and $\bar{N} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ obtained by projection on the Platonic screen of N' and N, respectively, are two nonrelativistic fields of observers related by a Milne boost parameterised by the 1-form $\bar{\chi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ defined as $\pi^*\bar{\chi} \equiv \chi$.*

Under a relativistic Milne boost parameterised by the 1-form χ, the following quantities transform as:

- $\overset{N}{A} \to \overset{N}{A} + \chi - \left(\chi(N) + \frac{1}{2}\overset{N}{h}(\chi,\chi)\right)\psi - \chi(\xi)\,\overset{N}{A}$
- $\overset{N}{\gamma} \to \overset{N}{\gamma} - \psi \otimes \chi - \chi \otimes \psi + 2\left(\chi(N) + \frac{1}{2}\overset{N}{h}(\chi,\chi)\right)\psi \otimes \psi + \chi(\xi)\left(\overset{N}{A} \otimes \psi + \psi \otimes \overset{N}{A}\right)$
- $\overset{N}{h} \to \overset{N}{h} - \overset{N}{h}(\chi) \otimes \xi - \xi \otimes \overset{N}{h}(\chi) + h(\chi,\chi)\,\xi \otimes \xi$

The principal objects introduced so far are summarised in the following Table:

Symbol	Type	Definition	Algebraic Properties
		Primary objects on $\mathscr{M}$	
g	$\Gamma\left(\vee^2 T^*\mathscr{M}\right)$		$\mathrm{Rad}\ g = 0$
ξ	$\Gamma(T\mathscr{M})$		$g(\xi,\xi) = 0$
N	$\Gamma(T\mathscr{M})$		$g(N,N) = 0,\ g(\xi,N) = 1$
		Secondary objects on $\mathscr{M}$	
ψ	$\Omega^1(\mathscr{M})$	$\psi \equiv g(\xi)$	$\psi(\xi) = 0,\ \psi(N) = 1$
$\overset{N}{A}$	$\Omega^1(\mathscr{M})$	$\overset{N}{A} \equiv g(N)$	$\overset{N}{A}(N) = 0,\ \overset{N}{A}(\xi) = 1$
$\overset{N}{h}$	$\Gamma\left(\vee^2 T\mathscr{M}\right)$	$\overset{N}{h} \equiv g^{-1} - 2N \vee \xi$	$\mathrm{Rad}\ \overset{N}{h} = \mathrm{Span}\left\{\psi, \overset{N}{A}\right\}$
$\overset{N}{\gamma}$	$\Gamma\left(\vee^2 T^*\mathscr{M}\right)$	$\overset{N}{\gamma} \equiv g - 2\psi \vee \overset{N}{A}$	$\mathrm{Rad}\ \overset{N}{\gamma} = \mathrm{Span}\{\xi, N\}$
		Induced objects on $\bar{\mathscr{M}}$	
$\bar{h}$	$\Gamma\left(\vee^2 T\bar{\mathscr{M}}\right)$	$\bar{h} \equiv \pi_* g^{-1}$	$\overset{N}{\bar{\gamma}}\left(\bar{h}(\bar{\alpha}), \bar{X}\right) = \bar{\alpha}\left(\bar{X}\right) - \bar{\psi}\left(\bar{X}\right)\bar{\alpha}\left(\bar{N}\right)$,
$\overset{N}{\bar{\gamma}}$	$\Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$	$\pi^*\overset{N}{\bar{\gamma}} \equiv \overset{N}{\gamma}$	$\forall X \in \Gamma\left(T\bar{\mathscr{M}}\right)$ and $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$
$\bar{\psi}$	$\Omega^1\left(\bar{\mathscr{M}}\right)$	$\pi^*\bar{\psi} = \psi$	$\bar{h}\left(\bar{\psi}\right) = 0$
$\bar{N}$	$\Gamma\left(T\bar{\mathscr{M}}\right)$	$\bar{N} \equiv \pi_* N$	$\overset{N}{\bar{\gamma}}\left(\bar{N}\right) = 0,\ \bar{\psi}\left(\bar{N}\right) = 1$

4.4 Projection of a Koszul connection

Having seen how nonrelativistic metric structures can be embedded inside Kundt waves, the next logical step consists in investigating how a notion of parallelism on a gravitational wave can be lowered downto its Platonic screen. Such a procedure is well-known in the case where the gravitational wave is a Bargmann-Eisenhart wave and the induced Koszul connection is then Newtonian (*cf.* [24]). In this Section, we review this construction of Duval *et al.* and then investigate potential generalisations to Platonic waves. We will focus on torsionfree connections, so that the ambient parallelism will be furnished by the Levi-Civita connection.

To formulate things in a concrete fashion, one is seeking for a prescription in order to define on the Platonic screen $\bar{\mathscr{M}}$ a Koszul connection $\bar{\nabla} : \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right) \to \Gamma\left(T\bar{\mathscr{M}}\right) : \bar{X} \times \bar{Y} \mapsto \bar{\nabla}_{\bar{X}}\bar{Y}$ where $\bar{X}$ and $\bar{Y}$ are two nonrelativistic vector fields. The most natural way in order to define such a nonrelativistic Koszul connection on $\bar{\mathscr{M}}$ from the Levi-Civita connection ∇ consists in first lifting the vector fields $\left(\bar{X}, \bar{Y}\right)$ on $\mathscr{M}$, then performing the Levi-Civita parallelism on the gravitational wave and finally projecting the obtained vector field back downto the Platonic screen $\bar{\mathscr{M}}$. The nonrelativistic Koszul connection is then defined by making the following diagram commute:

$$\begin{tikzcd}
(X, Y) \arrow[r, "\nabla"] \arrow[d, "\pi_*"] & \nabla_X Y \arrow[d, "\pi_*"] \\
\left(\bar{X}, \bar{Y}\right) \arrow[r, "\bar{\nabla}"] & \bar{\nabla}_{\bar{X}}\bar{Y}.
\end{tikzcd} \tag{4.4.18}$$

Of course, several conditions must be met in order for $\bar{\nabla}$ to be well defined:

1. The ambient vector field $\nabla_X Y \in \Gamma\left(T\mathscr{M}\right)$ must be projectable.
2. The nonrelativistic vector field $\bar{\nabla}_{\bar{X}}\bar{Y}$ must be independent of the choice of representatives X and Y.
3. The defined derivative operator $\bar{\nabla} : \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right) \to \Gamma\left(T\bar{\mathscr{M}}\right)$ must satisfy the axioms of a Koszul connection.

We now focus on Bargmann-Eisenhart waves and review the construction of [24] by spelling out all the details in order to prepare the subsequent generalisation.

4.4.1 Newtonian connection embedded in a Bargmann-Eisenhart wave

As far as Bargmann-Eisenhart waves are concerned, Condition 1 is insured for any couple of projectable vector fields, as expressed in the following Lemma:

Lemma 4.4.1. *Let $(\mathscr{M}, g, \xi)$ be a Bargmann-Eisenhart wave with associated Levi-Civita connection ∇. Let $X, Y \in \Gamma(T\mathscr{M})$ be two projectable vector fields. Then the vector field $\nabla_X Y \in \Gamma(T\mathscr{M})$ is projectable.*

Proof: Since $(\mathscr{M}, g, \xi)$ is assumed to be a Bargmann-Eisenhart wave, its wave vector field $\xi \in \Gamma(T\mathscr{M})$ is parallel with respect to the Levi-Civita connection ∇. In particular, $\xi \in \Gamma(T\mathscr{M})$ is Killing with respect to the metric g and then preserves the Levi-Civita connection ∇ (*cf.* Proposition A.9.10) so that $[\xi, \nabla_X Y] = \nabla_{[\xi,X]} Y + \nabla_X [\xi, Y]$, $\forall X, Y \in \Gamma(T\mathscr{M})$ (*cf.* Definition A.9.7). Furthermore, assuming that Y and Z are projectable, by Definition 4.1.8, there exist two functions $f, g \in C^\infty(\mathscr{M})$ such that $[\xi, X] = f\xi$ and $[\xi, Y] = g\xi$. Substituting, one obtains $[\xi, \nabla_X Y] = \nabla_{f\xi} Y + \nabla_X (g\xi) = f\nabla_\xi Y + X[g]\,\xi + g\nabla_X \xi$. The last term vanishes since ξ is parallel while the torsion-free condition allows to express the first term as $f\nabla_\xi Y = f\nabla_Y \xi + f[\xi, Y] = fg\xi$. Finally, the expression for $[\xi, \nabla_X Y]$ reads $[\xi, \nabla_X Y] = (fg + X[g])\,\xi$, so that $\nabla_X Y$ is seen to be projectable. $\square$

Having established that the vector field $\nabla_X Y$ admits a projection on the Platonic screen, we now investigate how this projected vector field depends on the choice of lift. Indeed, the following Lemma establishes its independence on the choice of representative lifts (Condition 2):

Lemma 4.4.2. *Let $(\mathscr{M}, g, \xi)$ be a Bargmann-Eisenhart wave with associated Levi-Civita connection ∇. Let $\bar{X}$ and $\bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ be two vector fields on the Platonic screen $\bar{\mathscr{M}}$ and $X, Y \in \Gamma(T\mathscr{M})$ be two arbitrary lifts for $\bar{X}$ and $\bar{Y}$, respectively. The vector field $\bar{\nabla}_{\bar{X}} \bar{Y} \equiv \pi_*(\nabla_X Y) \in \Gamma\left(T\bar{\mathscr{M}}\right)$ is independent of the choice of lifts X and Y.*

Proof: According to Proposition 4.1.11, two lifts X and X' of a vector field $\bar{X}$ differ by $X' - X = f\xi$, with $f \in C^\infty(\mathscr{M})$ a function on $\mathscr{M}$. This leads to $\nabla_{X'} Y = \nabla_X Y + f\nabla_\xi Y$. Since ∇ is torsionfree by hypothesis, it satisfies in particular $\nabla_\xi Y - \nabla_Y \xi - [\xi, Y] = 0$, $\forall Y \in \Gamma(T\mathscr{M})$. The second term vanishes while the third term can be rewritten as $[\xi, Y] = g\xi$, with $g \in C^\infty(\mathscr{M})$, since Y is projectable, being a lift. In the end, one obtains $\nabla_{X'} Y = \nabla_X Y + fg\xi$, so that $\pi_*(\nabla_{X'} Y) = \pi_*(\nabla_X Y + fg\xi) = \pi_*(\nabla_X Y)$ and $\bar{\nabla}_{\bar{X}} \bar{Y}$ is thus independent of the choice of lift for $\bar{X}$.

Similarly, two lifts Y and Y' of $\bar{Y}$ differ by $Y' \equiv Y + l\xi$, with $l \in C^\infty(\mathscr{M})$ so that one can compute $\nabla_X Y' = \nabla_X Y + Y[l]\,\xi + l\nabla_X \xi = \nabla_X Y + Y[l]\,\xi$ since ξ is parallelised by ∇. Applying π_* on both sides leads to $\pi_*(\nabla_X Y') = \pi_*(\nabla_X Y + Y[l]\,\xi) = \pi_*(\nabla_X Y)$, so that $\bar{\nabla}_{\bar{X}} \bar{Y}$ does not depend on the choice of lift for $\bar{Y}$. $\square$

Lemmas 4.4.1 and 4.4.2 together ensure that, in the case where the gravitational wave is a Bargmann-Eisenhart wave, the derivative operator $\bar{\nabla} : \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right) \to \Gamma\left(T\bar{\mathscr{M}}\right)$, obtained by making Diagram 4.4.18 commute, is well-defined. We now verify that it is a Koszul connection (Condition 3):

Proposition 4.4.3. *Let $(\mathscr{M}, g, \xi)$ be a Bargmann-Eisenhart wave with associated Levi-Civita connection ∇. The derivative operator $\bar{\nabla} : \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right) \to \Gamma\left(T\bar{\mathscr{M}}\right)$ defined by commutation of the following diagram*

$$\begin{tikzcd} (X,Y) \arrow[r, "\nabla"] \arrow[d, "\pi_*"] & \nabla_X Y \arrow[d, "\pi_*"] \\ \left(\bar{X},\bar{Y}\right) \arrow[r, "\bar{\nabla}"] & \bar{\nabla}_{\bar{X}}\bar{Y}. \end{tikzcd}$$

where X and Y are arbitrary lifts of the vector fields $\bar{X}$ and $\bar{Y}$, respectively, is a well-defined Koszul connection on the Platonic screen $\bar{\mathscr{M}}$.

Proof: We start by recalling the axioms of a Koszul connection, when acting on vector fields (*cf.* Definition A.9.1) :

1. $\bar{\nabla}_{\bar{X}}\left(\bar{Y}+\bar{Z}\right) = \bar{\nabla}_{\bar{X}}\bar{Y} + \bar{\nabla}_{\bar{X}}\bar{Z}$
2. $\bar{\nabla}_{\bar{X}+\bar{Y}}\bar{Z} = \bar{\nabla}_{\bar{X}}\bar{Z} + \bar{\nabla}_{\bar{Y}}\bar{Z}$
3. $\bar{\nabla}_{\bar{f}\bar{X}}\bar{Y} = \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$
4. $\bar{\nabla}_{\bar{X}}\left(\bar{f}\bar{Y}\right) = \bar{X}\left[\bar{f}\right]\bar{Y} + \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$.

with $\bar{X}, \bar{Y}$ and $\bar{Z} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ and $f \in C^\infty\left(\bar{\mathscr{M}}\right)$. Axioms 1 and 2 are straightforwardly established from the linearity of ∇, π_* and the lift of a vector field. As for Axiom 3, one makes use of the lift $f \in C^\infty\left(\mathscr{M}\right)$ of a function $\bar{f}$ (*cf.* Definition 4.1.6) to write: $\bar{\nabla}_{\bar{f}\bar{X}}\bar{Y} = \pi_*\left(\nabla_{fX}Y\right) = \pi_*\left(f\nabla_X Y\right) = \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$. Finally, writing $\bar{\nabla}_{\bar{X}}\left(\bar{f}\bar{Y}\right) = \pi_*\left(\nabla_X fY\right) = \pi_*\left(X\left[f\right]Y + f\nabla_X Y\right) = \bar{X}\left[\bar{f}\right]\bar{Y} + \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$, where one used that $X\left[f\right]$ is ξ-invariant for X a projectable vector field and f a ξ-invariant function ($\mathcal{L}_\xi\left(X\left[f\right]\right) = \mathcal{L}_\xi X\left[f\right] + X\left[\mathcal{L}_\xi f\right] = g\xi\left[f\right] = 0$, where $g \in C^\infty\left(\mathscr{M}\right)$ is defined as $\left[\xi, X\right] = g\xi$) ensures that Axiom 4 is satisfied. □

Proposition 4.4.4. *The curvature operator for the Levi-Civita connection of a Bargmann-Eisenhart wave induces the curvature operator for the connection $\bar{\nabla}$ as:*

$$\bar{R}\left(\bar{X},\bar{Y};\bar{Z}\right) = \pi_*\left(R\left(X,Y;Z\right)\right) \tag{4.4.19}$$

where X, Y and $Z \in \Gamma\left(T\mathscr{M}\right)$ are the respective lifts of the vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z} \in \Gamma\left(T\bar{\mathscr{M}}\right)$.

Proof: The curvature operator associated to the Levi-Civita connection ∇ acts on a vector field $Z \in \Gamma(T\mathscr{M})$ as

$$R(X,Y;Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

with X, Y two vector fields of $\mathscr{M}$. We first check that the vector field $R(X,Y;Z) \in \Gamma(T\mathscr{M})$ is projectable whenever X, Y and Z are projectable. Proposition 4.4.1 ensures that $\nabla_Y Z$ is projectable, whenever Y, Z are projectable. By iterating the reasoning, one concludes that $\nabla_X \nabla_Y Z$ (and similarly $\nabla_Y \nabla_X Z$) is also projectable. Moreover, the projectability of $\nabla_{[X,Y]} Z$ is ensured by Proposition 4.1.9 together with Proposition 4.4.1, so that $R(X,Y;Z)$ is projectable and its projection on the Platonic screen is straightforwardly given by:

$$\begin{aligned}
\pi_*(R(X,Y;Z)) &= \pi_*(\nabla_X \nabla_Y Z) - \pi_*(\nabla_Y \nabla_X Z) - \pi_*\left(\nabla_{[X,Y]} Z\right) \\
&= \bar{\nabla}_{\bar{X}}(\pi_*(\nabla_Y Z)) - \bar{\nabla}_{\bar{Y}}(\pi_*(\nabla_X Z)) - \bar{\nabla}_{[\bar{X},\bar{Y}]}(\pi_* Z) \\
&= \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{Z} - \bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{Z} - \bar{\nabla}_{[\bar{X},\bar{Y}]} \bar{Z}.
\end{aligned}$$

□

Before characterising further the induced Koszul connection $\bar{\nabla}$ on $\bar{\mathscr{M}}$ (namely as a Galilean then Newtonian connection), we establish the two following technical Lemmas:

Lemma 4.4.5. *Let $(\mathscr{M}, g, \xi)$ be a Bargmann-Eisenhart wave with Levi-Civita connection ∇. The 1-form $\nabla_X \alpha \in \Omega^1(\mathscr{M})$ is projectable whenever $X \in \Gamma(T\mathscr{M})$ and $\alpha \in \Omega^1(\mathscr{M})$ are projectable.*

Proof: Recall, by Definition 4.1.8, that there exists a function $f \in C^\infty(\mathscr{M})$ such that $[\xi, X] = f\xi$. Similarly, Definition 4.1.14 ensures that $\alpha(\xi) = 0$ and $\mathcal{L}_\xi \alpha = 0$. We now need to verify that $\nabla_X \alpha$ satisfy these two same conditions.

The first condition ($(\nabla_X \alpha)(\xi) = 0$) is seen to be met using expression (A.9.12) in order to compute $(\nabla_X \alpha)(\xi) = X[\alpha(\xi)] - \alpha(\nabla_X \xi)$. Both terms vanish as can be seen using that α is projectable and that ∇ parallelises ξ.

The second condition to be satisfied in order for $\nabla_X \alpha$ to be projectable is $\mathcal{L}_\xi(\nabla_X \alpha) = 0$. The Lie derivative of $\nabla_X \alpha$ can be decomposed as $\mathcal{L}_\xi(\nabla_X \alpha) = (\mathcal{L}_\xi \nabla)_X \alpha + \nabla_{[\xi,X]} \alpha + \nabla_X(\mathcal{L}_\xi \alpha)$. The first term on the right-hand side vanishes due to the fact that the wave vector field is ∇-preserving, being Killing. The last term vanishes as well since α is projectable. The second term can be modified using that X is projectable, as $\nabla_{[\xi,X]} \alpha = f \nabla_\xi \alpha$. We thus need to show that the 1-form $\nabla_\xi \alpha$ vanishes identically. Acting on a vector field $Y \in \Gamma(T\mathscr{M})$:

$$\begin{aligned}(\nabla_\xi \alpha)(Y) &= \xi[\alpha(Y)] - \alpha(\nabla_\xi Y)\\ &= \xi[\alpha(Y)] - \alpha(\nabla_Y \xi + [\xi, Y])\\ &= \xi[\alpha(Y)] - \alpha([\xi, Y])\\ &= (\mathcal{L}_\xi \alpha)(Y)\\ &= 0\end{aligned}$$

where one used respectively: in the first line expression (A.9.12), in the second line the torsionfree condition, in the third line the fact that ξ is parallelised by ∇, in the fourth line expression (4.1.2) and in the fifth line the fact that α is projectable. □

Lemma 4.4.6. *The following diagram commutes:*

$$\begin{tikzcd} \bar{\alpha} \arrow[r, "\pi^*"] \arrow[d, "\bar{\nabla}_{\bar{X}}"'] & \alpha \arrow[d, "\nabla_X"] \\ \bar{\nabla}_{\bar{X}}\bar{\alpha} \arrow[r, "\pi^*"] & \nabla_X \alpha \end{tikzcd}$$

with

- $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ *a 1-form on the Platonic screen*
- $\alpha \in \Omega^1(\mathscr{M})$ *the pullback of* $\bar{\alpha}$ *by the projection* π
- $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ *a vector field on the Platonic screen*
- $X \in \Gamma(T\mathscr{M})$ *a lift of* $\bar{X}$ *on* $\mathscr{M}$.

Proof: Since X and α are both projectable (being a lift and the pullback of a 1-form on $\bar{\mathscr{M}}$, respectively), Proposition 4.4.5 ensures that $\nabla_X \alpha$ is projectable, so that there exists a 1-form $\bar{\beta} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ such that $\pi^*\bar{\beta} = \nabla_X \alpha$. In order to show that $\bar{\beta} = \bar{\nabla}_{\bar{X}}\bar{\alpha}$, we let the pointwise 1-form $(\nabla_X \alpha)_x$ acts on a vector $Y_x \in T_x\mathscr{M}$, with $x \in \mathscr{M}$, and perform the straightforward manipulation (the following expressions are all pointwise but we dropped the point indices for notational simplicity):

$$\begin{aligned}(\nabla_X \alpha)(Y) &= X[\alpha(Y)] - \alpha(\nabla_X Y)\\ &= X[(\pi^*\bar{\alpha})(Y)] - (\pi^*\bar{\alpha})(\nabla_X Y)\\ &= X[\bar{\alpha}(\pi_* Y)] - \bar{\alpha}(\pi_* \nabla_X Y)\\ &= (\pi_* X)[\bar{\alpha}(\pi_* Y)] - \bar{\alpha}\left(\bar{\nabla}_{\bar{X}}(\pi_* Y)\right)\\ &= \left(\bar{X}[\bar{\alpha}(\pi_* Y)] - \bar{\alpha}\left(\bar{\nabla}_{\bar{X}}(\pi_* Y)\right)\right)\\ &= \left(\bar{\nabla}_{\bar{X}}\bar{\alpha}\right)(\pi_* Y)\\ &= \pi^*\left(\bar{\nabla}_{\bar{X}}\bar{\alpha}\right)(Y)\end{aligned}$$

where in the first and sixth line equality (A.9.12) has been used as well as expression (A.2.5) in the third and seventh line. □

Proposition 4.4.7. *The induced Koszul connection $\bar{\nabla}$ is torsionfree and compatible with the nonrelativistic metric $\bar{h}$ and the absolute clock $\bar{\psi}$.*

Proof: We first check that $\bar{\nabla}$ is torsionfree and then its compatibility with the metric structure $\left(\bar{\psi}, \bar{h}\right)$. These features will be seen to descend in a straightforward fashion from the properties of the Levi-Civita connection on $\mathscr{M}$. Indeed, the torsionfree condition enjoyed by ∇ is passed straightforwardly on $\bar{\nabla}$. Explicitly, this condition reads: $\nabla_X Y - \nabla_Y X - [X, Y] = 0$, for all $X, Y \in \Gamma\left(T\mathscr{M}\right)$. Assuming that X and Y are projectable and acting with π_* on both sides leads to

$$\begin{aligned} \pi_*\left(\nabla_X Y\right) - \pi_*\left(\nabla_Y X\right) - \pi_*\left[X, Y\right] &= 0 \\ \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X} - \left[\bar{X}, \bar{Y}\right] &= 0 \end{aligned}$$

with $\bar{X} \equiv \pi_* X$ and $\bar{Y} \equiv \pi_* Y$. Proposition 4.1.12 has been used to modify the third term.

In order to prove the compatibility of $\bar{\nabla}$ with the contravariant nonrelativistic metric $\bar{h}$ (*cf.* Section 4.2), we start from the metric compatibility of ∇, expressed as $X\left[g^{-1}\left(\alpha, \beta\right)\right] = g^{-1}\left(\nabla_X \alpha, \beta\right) + g^{-1}\left(\alpha, \nabla_X \beta\right)$, with $X \in \Gamma\left(T\mathscr{M}\right)$ and $\alpha, \beta \in \Omega^1\left(\mathscr{M}\right)$. One now makes the assumption that X is a lift for the vector field $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ while α and β are assumed to be projectable. Hence, there exist two 1-forms $\bar{\alpha}, \bar{\beta} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ on the Platonic screen such that $\pi^*\bar{\alpha} = \alpha$ and $\pi^*\bar{\beta} = \beta$. Now, according to Lemma 4.4.6, the 1-form $\nabla_X \alpha$ can be expressed as $\nabla_X \alpha = \pi^*\left(\bar{\nabla}_{\bar{X}}\bar{\alpha}\right)$, and similarly $\nabla_X \beta = \pi^*\left(\bar{\nabla}_{\bar{X}}\bar{\beta}\right)$.
By definition of $\bar{h}$ as the generalised pushforward of g^{-1}, the identity

$$\bar{h}\left(\bar{\omega}_1, \bar{\omega}_2\right) \circ \pi = g^{-1}\left(\omega_1, \omega_2\right)$$

holds for all projectable 1-forms $\omega_1, \omega_2 \in \Omega^1\left(\mathscr{M}\right)$ with respective projections $\bar{\omega}_1, \bar{\omega}_2 \in \Omega^1\left(\bar{\mathscr{M}}\right)$. We now make use of this identity in order to modify the expression for the metric compatibility condition of ∇ as follows:

$$\begin{aligned} X\left[g^{-1}\left(\alpha, \beta\right)\right] &= g^{-1}\left(\nabla_X \alpha, \beta\right) + g^{-1}\left(\alpha, \nabla_X \beta\right) \\ X\left[\bar{h}\left(\bar{\alpha}, \bar{\beta}\right) \circ \pi\right] &= \bar{h}\left(\bar{\nabla}_{\bar{X}}\bar{\alpha}, \bar{\beta}\right) \circ \pi + \bar{h}\left(\bar{\alpha}, \bar{\nabla}_{\bar{X}}\bar{\beta}\right) \circ \pi \\ \bar{X}\left[\bar{h}\left(\bar{\alpha}, \bar{\beta}\right)\right] \circ \pi &= \bar{h}\left(\bar{\nabla}_{\bar{X}}\bar{\alpha}, \bar{\beta}\right) \circ \pi + \bar{h}\left(\bar{\alpha}, \bar{\nabla}_{\bar{X}}\bar{\beta}\right) \circ \pi \\ \bar{X}\left[\bar{h}\left(\bar{\alpha}, \bar{\beta}\right)\right] &= \bar{h}\left(\bar{\nabla}_{\bar{X}}\bar{\alpha}, \bar{\beta}\right) + \bar{h}\left(\bar{\alpha}, \bar{\nabla}_{\bar{X}}\bar{\beta}\right) \end{aligned}$$

where in the last step, the surjectivity of π has been used.
Regarding the compatibility of $\bar{\nabla}$ with $\bar{\psi}$, we first establish the following Lemma:
Lemma 4.4.8. *The wave covector field $\psi \equiv g(\xi) \in \Omega^1(\mathscr{M})$ of a Bargmann-Eisenhart wave $(\mathscr{M}, g, \xi)$ is preserved by the Levi-Civita connection ∇.*

Proof: By definition, the wave covector field ψ will be preserved by ∇ if and only if $X[\psi(Y)] = \psi(\nabla_X Y)$, $\forall X, Y \in \Gamma(T\mathscr{M})$. Using the metric compatibility of ∇ (*cf.* Proposition A.9.4), one writes $X[\psi(Y)] = X[g(\xi, Y)] = g(\nabla_X \xi, Y) + g(\xi, \nabla_X Y) = \psi(\nabla_X Y)$, since ξ is parallelised by ∇. □

The previous Lemma thus establishes the compatibility of ∇ with the wave covector field ψ, expressed as $X[\psi(Y)] = \psi(\nabla_X Y)$, where X and $Y \in \Gamma(T\mathscr{M})$ are now assumed to be lifts of the vector fields $\bar{X}$ and $\bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$, respectively. Using the expression of the absolute clock $\bar{\psi}$ (*cf.* Definition 4.1.17) and manipulating leads to:

$$\begin{aligned}
X\left[\pi^*\bar{\psi}(Y)\right] &= \pi^*\bar{\psi}(\nabla_X Y) \\
X\left[\bar{\psi}(\pi_* Y)\circ\pi\right] &= \bar{\psi}(\pi_*\nabla_X Y)\circ\pi \\
\bar{X}\left[\bar{\psi}\left(\bar{Y}\right)\right]\circ\pi &= \bar{\psi}\left(\bar{\nabla}_{\bar{X}}\bar{Y}\right)\circ\pi \\
\bar{X}\left[\bar{\psi}\left(\bar{Y}\right)\right] &= \bar{\psi}\left(\bar{\nabla}_{\bar{X}}\bar{Y}\right).
\end{aligned}$$

where expression (A.2.6) has been used in the first step, along with the surjectivity of π in the last step. □

The nonrelativistic Koszul connection induced on the Platonic screen of a Bargmann-Eisenhart wave is then a torsionfree Galilean connection. Consistency with Proposition 3.2.20 requires the nonrelativistic metric structure embedded in a Bargmann-Eisenhart wave to be Augustinian (*i.e.* with closed absolute clock $\bar{\psi}$). Indeed, it has been shown in Proposition 4.1.19 that the metric structure induced by a Platonic wave is Augustinian if and only if the wave belongs to the Bargmann-Eisenhart class. A direct consequence of this result, together with Proposition 3.2.20 is the following fact: the torsionfree nonrelativistic Koszul connection induced on the Platonic screen of a proper Platonic wave (*i.e.* not a Bargmann-Eisenhart wave) cannot be Galilean. We then foresee that Platonic waves must induce new notions of parallelism on their Platonic screen.

We conclude this Section by the following Proposition:

Proposition 4.4.9. *The induced Koszul connection on $\bar{\mathscr{M}}$ satisfies the Duval-Künzle condition.*

Proof: One starts from the symmetry relation of the Levi-Civita curvature A.9.18 *i.e.* $R(X, Y; Z, W) = R(Z, W; X, Y)$ with $X, Y, Z, W \in \Gamma(T\mathscr{M})$ four vector fields. The

1-forms dual to W and Y are denoted α and $\beta \in \Omega^1(\mathscr{M})$, respectively, so that the symmetry relation can be equivalently expressed as

$$\alpha\left(R\left(X, g^{-1}(\beta); Z\right)\right) = \beta\left(R\left(Z, g^{-1}(\alpha); X\right)\right). \tag{4.4.20}$$

We make the assumption that X and Z are projectable vector fields with $\pi_* X \equiv \bar{X}$ and $\pi_* Z \equiv \bar{Z}$. We suppose furthermore that α and β are projectable 1-forms, with $\alpha \equiv \pi^* \bar{\alpha}$ and $\beta \equiv \pi^* \bar{\beta}$, respectively (equivalently, Y and W can be assumed to be orthogonal to the wave vector field ξ and ξ-invariant, *cf.* Proposition 4.1.15). Using these equalities as well as Proposition 4.4.4, the curvature symmetry relation (4.4.20) can be expressed as:

$$\bar{\alpha}\left(\bar{R}\left(\bar{X}, \pi_*\left(g^{-1}\beta\right); \bar{Z}\right)\right) \circ \pi = \bar{\beta}\left(\bar{R}\left(\bar{Z}, \pi_*\left(g^{-1}(\alpha); \bar{X}\right)\right)\right) \circ \pi.$$

We now prove the following technical Lemma:

Lemma 4.4.10. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $\bar{\omega} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ be a 1-form on the Platonic screen and denote $\omega \equiv \pi^* \bar{\omega}$ its pullback by the projection π on the Platonic wave $\mathscr{M}$. The following relation holds:*

$$\pi_*\left(g^{-1}(\omega)\right) = \bar{h}(\bar{\omega}). \tag{4.4.21}$$

Proof: Proposition 4.1.15 guarantees that the vector field $g^{-1}(\omega)$ is ξ-invariant, so that the left-hand side of expression (4.4.21) is well-defined. One introduces a 1-form $\bar{\chi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ and writes $\bar{\chi}\left(\pi_*\left(g^{-1}(\omega)\right)\right) \circ \pi = \pi^* \bar{\chi}\left(g^{-1}(\omega)\right) = \chi\left(g^{-1}(\omega)\right) = g^{-1}(\chi, \omega)$, where $\chi \equiv \pi^* \bar{\chi}$. By definition of $\bar{h}$, this leads to $\bar{\chi}\left(\pi_*\left(g^{-1}(\omega)\right)\right) \circ \pi = g^{-1}(\pi^* \bar{\chi}, \pi^* \bar{\omega}) = \left(\pi_* g^{-1}\right)(\bar{\chi}, \bar{\omega}) \circ \pi = \bar{h}(\bar{\chi}, \bar{\omega}) \circ \pi = \bar{\chi}\left(\bar{h}(\bar{\omega})\right) \circ \pi$, $\forall \bar{\chi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ and the expected relation is then seen to hold. □

Our symmetry relation now takes the form

$$\bar{\alpha}\left(\bar{R}\left(\bar{X}, \bar{h}\left(\bar{\beta}\right); \bar{Z}\right)\right) \circ \pi = \bar{\beta}\left(\bar{R}\left(\bar{Z}, \bar{h}(\bar{\alpha}); \bar{X}\right)\right) \circ \pi$$

which, thanks to the surjectivity of π, guarantees that expression 3.2.33 holds. □

We sum up this Section and gather Propositions 4.4.3, 4.4.7 and 4.4.9 in the following important Theorem:

Theorem 4.4.11 (Duval *et al.* [24])**.** *The Platonic screen of a Bargmann-Eisenhart wave is a Newtonian manifold.*

This Theorem can be formulated in components by expressing the Christoffel symbols for the Levi-Civita connection of a Bargmann-Eisenhart wave (*cf.* eq.(3.1.1))

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

in terms of objects admitting a well-defined projection on the Platonic screen. We recall from Section 4.2 that this is not the case of the contravariant metric g which must be reexpressed using an arbitrary (projectable) relativistic field of light-like observers $N \in FO(\mathscr{M})$ as:

$$g_{\mu\nu} = \overset{N}{\gamma}_{\mu\nu} + \psi_\mu \overset{N}{A}_\nu + \psi_\nu \overset{N}{A}_\mu \tag{4.4.22}$$

where $\overset{N}{A} \equiv g(N) \in \Omega^1(\mathscr{M})$ is the dual 1-form associated to N. The Christoffel symbols take then the form:

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} \;=\;& \xi^\lambda \partial_{(\mu} \overset{N}{A}_{\nu)} + N^\lambda \partial_{(\mu}\psi_{\nu)} + \frac{1}{2} g^{\lambda\rho}\left[\partial_\mu \overset{N}{\gamma}_{\rho\nu} + \partial_\nu \overset{N}{\gamma}_{\rho\mu} - \partial_\rho \overset{N}{\gamma}_{\mu\nu}\right] + g^{\lambda\rho}\psi_{(\mu} \overset{N}{F}_{\nu)\rho} \\ & + g^{\lambda\rho} \overset{N}{A}_\mu \partial_{[\nu}\psi_{\rho]} + g^{\lambda\rho} \overset{N}{A}_\nu \partial_{[\mu}\psi_{\rho]}\end{aligned} \tag{4.4.23}$$

with $\overset{N}{F} \equiv d\overset{N}{A} \in \Omega^2(\mathscr{M})$ the gravitational field strength, or curvature, associated to $\overset{N}{A}$. The first term, being along the wave vector field ξ, is annihilated by projection while the two terms appearing in the second line vanish due to the closedness of the wave covector field of a Bargmann-Eisenhart wave (*cf.* Proposition 4.1.19). All the remaining terms project well on the Platonic screen so that the Christoffel symbols of a Bargmann-Eisenhart wave project downto the components of a Newtonian connection (*cf.* eq.(3.2.20)):

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \bar{N}^\lambda \partial_{(\mu}\bar{\psi}_{\nu)} + \frac{1}{2}\bar{h}^{\lambda\rho}\left[\partial_\mu \overset{\bar{N}}{\bar{\gamma}}_{\rho\nu} + \partial_\nu \overset{\bar{N}}{\bar{\gamma}}_{\rho\mu} - \partial_\rho \overset{\bar{N}}{\bar{\gamma}}_{\mu\nu}\right] + \bar{h}^{\lambda\rho}\bar{\psi}_{(\mu} \overset{\bar{N}}{\bar{F}}_{\nu)\rho} \tag{4.4.24}$$

where

$$\begin{cases} \bar{N} \equiv \pi_* N \\ \bar{h} \equiv \pi_* g^{-1} \\ \pi^* \bar{\psi} \equiv \psi \\ \pi^* \overset{\bar{N}}{\bar{\gamma}} \equiv \overset{N}{\gamma} \\ \pi^* \overset{\bar{N}}{\bar{F}} \equiv \overset{N}{F} \end{cases}$$

and the closedness of $\bar{\psi}$ and $\overset{\bar{N}}{\bar{F}}$ follow from that of ψ and $\overset{N}{F}$, respectively. Note that the 1-form $\overset{N}{A}$ dual to N does not project on the Platonic screen (since $\overset{N}{A}(\xi) = 1$). However, introducing a section $\sigma : \bar{\mathscr{U}} \subset \bar{\mathscr{M}} \to \mathscr{M}$ (with $\bar{\mathscr{U}}$ an open subset of $\bar{\mathscr{M}}$) allows to define the 1-form $\overset{\bar{N}}{A} \in \Omega^1\left(\bar{\mathscr{U}}\right)$ as $\overset{\bar{N}}{A} \equiv \sigma^* \overset{N}{A}$. Whenever N is ξ-invariant, $\overset{N}{A}$ is an Ehresmann connection and $\overset{\bar{N}}{A}$ a gauge connection on $\bar{\mathscr{U}}$. Under a change of section, $\overset{\bar{N}}{A}$ changes as $\overset{\bar{N}}{A} \to \overset{\bar{N}}{A} + df$ where one recognises the Maxwell gauge-transformation of Section 3.2.2. Furthermore, one can make use of a choice of section in order to define a Lagrangian metric $\bar{g} \in \Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$ as $\bar{g} \equiv \sigma^* g$. Eq.(4.4.22) then accounts for the expression $\bar{g} = \overset{\bar{N}}{\gamma} + 2\bar{\psi} \vee \overset{\bar{N}}{A}$ (*cf.* Table 3.1).

Note that the above construction is independent of the choice of relativistic field of light-like observers N. Indeed, it can be checked that the coefficients (4.4.23) and (4.4.24) are respectively invariant under a relativistic and nonrelativistic Milne boost.

4.4.2 Kundt connection on the absolute spaces

Before generalising the results of the precedent Section to Platonic waves (*cf.* Section 4.4.3), we devote the present Section to the investigation of the notion of parallelism induced by Kundt waves on their absolute spaces. Concretely, we provide a proof of Proposition 4.4.16 which asserts that the Levi-Civita connection of a Kundt wave $(\mathscr{M}, g, \xi)$ projects onto the absolute spaces of the Platonic screen $\bar{\mathscr{M}}$ as the Levi-Civita connection associated to the spatial metric $\bar{\gamma}$. We start with a technical Lemma before showing that the three consistency conditions below Diagram 4.4.18 are satisfied when restricting to spatial vector fields on $\bar{\mathscr{M}}$.

Lemma 4.4.12. *Let $(\mathscr{M}, g, \xi)$ be a gravitational wave with Levi-Civita connection ∇ and let $X, Y \in \Gamma(T\mathscr{M})$ be a pair of ξ-orthogonal vector fields. The following Propositions hold if and only if $(\mathscr{M}, g, \xi)$ is a Kundt wave:*

1. $g(\nabla_X \xi, Y) = 0$
2. $(\mathcal{L}_\xi g)(X, Y) = 0$

Now, letting $(\mathscr{M}, g, \xi)$ be a Kundt wave:

3. *The vector field $\nabla_X Y \in \Gamma(T\mathscr{M})$ is ξ-orthogonal.*
4. *The vector field $\nabla_X \xi \in \Gamma(T\mathscr{M})$ is colinear to the wave vector field ξ.*
5. *If X is projectable, then the vector field $\nabla_\xi X \in \Gamma(T\mathscr{M})$ is colinear to the wave vector field ξ.*
6. *If the 1-form $\alpha \in \Omega^1(\mathscr{M})$ is the lift of a 1-form $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$, then $(\nabla_X \alpha)(\xi) = 0$.*

Proof:

1. Kundt waves are defined, among gravitational waves, by the vanishing of their three optical scalars (*cf.* Section 2.2.4), which can be expressed as

$$\left(\nabla_{P^N(V)}\psi\right)\left(P^N(W)\right) = 0 \tag{4.4.25}$$

for all vector fields $V, W \in \Gamma(T\mathscr{M})$ and for all observer $N \in FO(\mathscr{M})$ (*cf.* Definition 4.3.3 of the spacelike projection operator P^N). Restricting to ξ-orthogonal vector fields, the 1-form $\nabla_{P^N(X)}\psi \in \Omega^1(\mathscr{M})$ develops as

$$\nabla_{P^N(X)}\psi = \nabla_X\psi - \overset{N}{A}(X)\,\nabla_\xi\psi$$

where the second term vanishes since ξ is affine geodesic (*cf.* Proposition 2.1.4). Now, one is able to reformulate eq.(4.4.25) for ξ-orthogonal vector fields as:

$$\begin{aligned}\left(\nabla_{P^N(X)}\psi\right)\left(P^N(Y)\right) = 0 \quad &\Leftrightarrow \quad (\nabla_X\psi)\left(P^N(Y)\right) = 0\\ &\Leftrightarrow \quad (\nabla_X\psi)(Y) - \overset{N}{A}(Y)(\nabla_X\psi)(\xi) = 0\\ &\Leftrightarrow \quad g(\nabla_X\xi, Y) = 0\end{aligned}$$

where the second term on the second line vanishes since ξ is null.

2. Symmetrising the previously established identity in X and Y leads to the Killing equation of Proposition A.9.9, so that ξ is Killing when acting on ξ-orthogonal vector fields. Furthermore, the antisymmetric part

$$\frac{1}{2}\left[g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X)\right] = d\psi(X, Y)$$

vanishes whenever $X, Y \in \text{Ker}\ \psi$ (since ψ satisfies the Frobenius Criterion, $d\psi = \alpha \wedge \psi$ for some 1-form $\alpha \in \Omega^1(\mathscr{M})$ due to) so that $\mathcal{L}_\xi(X, Y) = 0 \Rightarrow g(\nabla_X\xi, Y) = 0$. Proposition 1 then ensures that the equivalence stands if and only if $(\mathscr{M}, g, \xi)$ is a Kundt wave.

3. The metric compatibility of ∇ allows to write $X[g(\xi, Y)] = g(\nabla_X\xi, Y) + g(\xi, \nabla_X Y)$. The left-hand side vanishes since Y is assumed to be ξ-orthogonal while the first term of the right-hand side cancels due to Proposition 1 of the present Lemma. The only term remaining is $g(\xi, \nabla_X Y) = 0$, so that $\nabla_X Y$ is ξ-orthogonal.

4. Using again the metric compatibility of ∇, one writes

$$X\left[g\left(\xi,\nabla_X\xi\right)\right]=g\left(\nabla_X\xi,\nabla_X\xi\right)+g\left(\xi,\nabla_X\nabla_X\xi\right). \tag{4.4.26}$$

   Since the vector fields X and ξ are both ξ-orthogonal, Proposition 3 ensures that $\nabla_X\xi$ is ξ-orthogonal, so that the left-hand side of eq.(4.4.26) vanishes. Furthermore, X and $\nabla_X\xi$ being ξ-orthogonal, the vector field $\nabla_X\nabla_X\xi$ is also ξ-orthogonal so that the second term of the right-hand side cancels. The vector field $\nabla_X\xi$ is then seen to be null and ξ-orthogonal, so that it is colinear to ξ.

5. The torsionfree condition of ∇ allows to write $\nabla_X\xi-\nabla_\xi X-[X,\xi]=0$. Using Proposition 4 and the fact that X is projectable (so that there exists a function $f\in C^\infty(\mathscr{M})$ such that $[\xi,X]=f\xi$) ensures that $\nabla_\xi X$ is along ξ.

6. Using expression (A.9.12) allows to write

$$\begin{aligned}\left(\nabla_X\alpha\right)(\xi) &= X\left[\alpha\left(\xi\right)\right]-\alpha\left(\nabla_X\xi\right)\\ &= X\left[\pi^*\bar{\alpha}\left(\xi\right)\right]-\pi^*\alpha\left(\nabla_X\xi\right)\\ &= X\left[\alpha\left(\pi_*\xi\right)\circ\pi\right]-\alpha\left(\pi_*\nabla_X\xi\right)\circ\pi\\ &= 0\end{aligned}$$

   where in the last step Proposition 4 has been used.

□

Lemma 4.4.13. *Let $(\mathscr{M},g,\xi)$ be a Kundt wave with Levi-Civita connection ∇ and let $X,Y\in\Gamma(T\mathscr{M})$ be a pair of projectable ξ-orthogonal vector fields. Then the vector field $\nabla_XY\in\Gamma(T\mathscr{M})$ is projectable.*

Proof: According to Proposition 2 of Lemma 4.4.12, the wave vector field ξ is Killing when acting on ξ-orthogonal vector fields so that it is also ∇-preserving, by Proposition A.9.10 *i.e.*

$$[\xi,\nabla_XY]=\nabla_{[\xi,X]}Y+\nabla_X[\xi,Y]. \tag{4.4.27}$$

Since X and Y are assumed to be projectable, there exist two functions $f,l\in C^\infty(\mathscr{M})$ such that $[\xi,X]=f\xi$ and $[\xi,Y]=l\xi$, so that eq.(4.4.27) can be reformulated as: $[\xi,\nabla_XY]=\nabla_{f\xi}Y+\nabla_X(l\xi)$ where both terms on the right-hand side can be seen to be along ξ, using Propositions 5 and 4 of Lemma 4.4.12, respectively. □

Lemma 4.4.14. *Let* $(\mathscr{M}, g, \xi)$ *be a Kundt wave with Levi-Civita connection* ∇ *and* $\bar{X}, \bar{Y} \in \operatorname{Ker} \bar{\psi}$ *two spacelike vector fields on the Platonic screen* $\bar{\mathscr{M}}$*. The vector field* $\pi_* (\nabla_X Y) \in \Gamma\left(T\bar{\mathscr{M}}\right)$ *with* $X, Y \in \Gamma(T\mathscr{M})$ *two lifts of* $\bar{X}$ *and* $\bar{Y}$*, respectively, is independent of the choice of lifts* X *and* Y*.*

Proof: Since $\bar{X}$ and $\bar{Y}$ are assumed to be spacelike vector fields, their respective lifts X and Y are necessarily ξ-orthogonal, so that Proposition 6 of Lemma 4.4.12 ensures that $\nabla_X Y$ is projectable and the vector field $\pi_* (\nabla_X Y)$ is then well-defined.
According to Proposition 4.1.11, two lifts X and X' of the vector field $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ differ by $X' - X = f\xi$, with $f \in C^\infty(\mathscr{M})$ a function on $\mathscr{M}$. Inserting into $\nabla_{X'} Y$ leads to the equality $\nabla_{X'} Y = \nabla_X Y + f \nabla_\xi Y$ where the second term on the right-hand side is guaranteed to be along ξ by Proposition 5 of Lemma 4.4.12, so that $\pi_* \nabla_{X'} Y = \pi_* \nabla_X Y$.
Similarly, two lifts Y and Y' of $\bar{Y}$ differ by $Y' \equiv Y + l\xi$, with $l \in C^\infty(\mathscr{M})$ so that one can compute $\nabla_X Y' = \nabla_X Y + Y[l]\,\xi + l \nabla_X \xi$ where the third term on the right-hand side is colinear to the wave vector field ξ according to Proposition 4 of Lemma 4.4.12. Applying π_* on both sides leads to $\pi_* (\nabla_X Y') = \pi_* \nabla_X Y$, so that the projection $\pi_* \nabla_X Y$ is independent of the choice of lift for $\bar{X}$ and $\bar{Y}$. □

We conclude from the previous Lemma that the Kundt Levi-Civita connection projects well (*i.e.* without the need of additional prescription) on the absolute spaces of the Platonic screen $\bar{\mathscr{M}}$. The following Proposition ensures furthermore that the induced derivative operator is a Koszul connection:

Lemma 4.4.15. *Let* $(\mathscr{M}, g, \xi)$ *be a Kundt wave with associated Levi-Civita connection* ∇*. The derivative operator* $\bar{\nabla} : \Gamma\left(\operatorname{Ker} \bar{\psi}\right) \times \Gamma\left(\operatorname{Ker} \bar{\psi}\right) \to \Gamma\left(\operatorname{Ker} \bar{\psi}\right)$ *defined by commutation of the following diagram*

$$\begin{tikzcd}
(X,Y) \arrow[r, "\nabla"] \arrow[d, "\pi_*"] & \nabla_X Y \arrow[d, "\pi_*"] \\
\left(\bar{X},\bar{Y}\right) \arrow[r, "\bar{\nabla}"] & \bar{\nabla}_{\bar{X}} \bar{Y}.
\end{tikzcd}$$

where X *and* Y *are arbitrary lifts of the spacelike vector fields* $\bar{X}$ *and* $\bar{Y}$*, respectively, is a well-defined Koszul connection on the absolute spaces of the Platonic screen.*

Proof: We start by recalling the axioms of a Koszul connection, when acting on vector fields (*cf.* Definition A.9.1) :

1. $\bar{\nabla}_{\bar{X}} \left(\bar{Y} + \bar{Z}\right) = \bar{\nabla}_{\bar{X}} \bar{Y} + \bar{\nabla}_{\bar{X}} \bar{Z}$

2. $\bar{\nabla}_{\bar{X}+\bar{Y}}\bar{Z} = \bar{\nabla}_{\bar{X}}\bar{Z} + \bar{\nabla}_{\bar{Y}}\bar{Z}$

3. $\bar{\nabla}_{\bar{f}\bar{X}}\bar{Y} = \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$

4. $\bar{\nabla}_{\bar{X}}\left(\bar{f}\bar{Y}\right) = \bar{X}\left[\bar{f}\right]\bar{Y} + \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$

with $\bar{X}, \bar{Y}$ and $\bar{Z} \in \Gamma\left(\text{Ker } \bar{\psi}\right)$ and $f \in C^{\infty}\left(\bar{\mathscr{M}}\right)$. Note that, since Ker $\bar{\psi}$ is a subbundle of the tangent bundle $T\bar{\mathscr{M}}$, $\bar{X}$, $\bar{Y} \in$ Ker $\bar{\psi} \Rightarrow f\bar{X} + g\bar{Y} \in$ Ker $\bar{\psi}$ for all functions $f, g \in C^{\infty}\left(\bar{\mathscr{M}}\right)$. Axioms 1 and 2 are then straightforwardly established from the linearity of ∇, π_* and the lift of a vector field. As for Axiom 3, one makes use of the lift $f \in C^{\infty}\left(\mathscr{M}\right)$ of a function $\bar{f}$ (*cf.* Definition 4.1.6) in order to write: $\bar{\nabla}_{\bar{f}\bar{X}}\bar{Y} = \pi_*\left(\nabla_{fX}Y\right) = \pi_*\left(f\nabla_X Y\right) = \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$. Finally, writing $\bar{\nabla}_{\bar{X}}\left(\bar{f}\bar{Y}\right) = \pi_*\left(\nabla_X fY\right) = \pi_*\left(X\left[f\right]Y + f\nabla_X Y\right) = \bar{X}\left[\bar{f}\right]\bar{Y} + \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$, where one used that $X\left[f\right]$ is ξ-invariant for X projectable and f ξ-invariant ($\mathcal{L}_\xi\left(X\left[f\right]\right) = \left(\mathcal{L}_\xi X\right)\left[f\right] + X\left[\mathcal{L}_\xi f\right] = l\xi\left[f\right] = 0$, where $\left[\xi, X\right] = l\xi$), ensures that Axiom 4 is satisfied. □

Proposition 4.4.16. *Let $\left(\mathscr{M}, g, \xi\right)$ be a Kundt wave with associated Levi-Civita connection ∇. The Koszul connection $\bar{\nabla} : \Gamma\left(\text{Ker } \bar{\psi}\right) \times \Gamma\left(\text{Ker } \bar{\psi}\right) \to \Gamma\left(\text{Ker } \bar{\psi}\right)$ defined (cf. Proposition 4.4.15) on the absolute spaces of the Platonic screen $\bar{\mathscr{M}}$ is the Levi-Civita connection associated to the Riemannian relativistic spatial metric $\bar{\gamma} \in \Gamma\left(\vee^2 T^*\bar{\mathscr{M}}\right)$ (cf. Definition 4.2.3).*

Proof: Our starting point is the Koszul formula for the Levi-Civita connection associated to the relativistic metric $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$:

$$\begin{aligned} g\left(\nabla_X Y, Z\right) &= \frac{1}{2}\Big(X\left[g\left(Y,Z\right)\right] + Y\left[g\left(X,Z\right)\right] - Z\left[g\left(X,Y\right)\right] \\ &+ g\left(\left[X,Y\right],Z\right) - g\left(\left[X,Z\right],Y\right) - g\left(\left[Y,Z\right],X\right)\Big). \end{aligned}$$

We now make the assumption that X, Y and $Z \in \Gamma\left(T\mathscr{M}\right)$ are projectable and ξ-orthogonal. From Proposition 3 of Lemma 4.4.12, one concludes that $\nabla_X Y$ is ξ-orthogonal while Lemma 4.4.14 ensures that it is projectable. By Proposition 4.4.15, we then have $\pi_*\left(\nabla_X Y\right) = \bar{\nabla}_{\bar{X}}\bar{Y}$, where $\bar{X} \equiv \pi_* X \in$ Ker $\bar{\psi}$ and $\bar{Y} \equiv \pi_* Y \in$ Ker $\bar{\psi}$ are the respective projection of X and Y on the Platonic screen $\bar{\mathscr{M}}$. Furthermore, the Lie bracket $\left[X, Y\right]$ is ξ-orthogonal whenever X and Y are ξ-orthogonal, due to the involutivity of the distribution Ker ψ. Furthermore, $\left[X, Y\right]$ is projectable according to Proposition 4.1.9 and satisfies furthermore $\pi_*\left[X, Y\right] = \left[\bar{X}, \bar{Y}\right]$ (*cf.* Proposition 4.1.12). Now, using these different results as well as Definition 4.2.3, one can reformulate the previous Koszul formula as:

$$\begin{aligned}\bar{\gamma}\left(\bar{\nabla}_{\bar{X}}\bar{Y},\bar{Z}\right)\circ\pi &= \frac{1}{2}\Big(\bar{X}\left[\bar{\gamma}\left(\bar{Y},\bar{Z}\right)\right]\circ\pi+\bar{Y}\left[\bar{\gamma}\left(\bar{X},\bar{Z}\right)\right]\circ\pi-\bar{Z}\left[\bar{\gamma}\left(\bar{X},\bar{Y}\right)\right]\circ\pi \\ &+ \bar{\gamma}\left(\left[\bar{X},\bar{Y}\right],\bar{Z}\right)\circ\pi-\bar{\gamma}\left(\left[\bar{X},\bar{Z}\right],\bar{Y}\right)\circ\pi-\bar{\gamma}\left(\left[\bar{Y},\bar{Z}\right],\bar{X}\right)\circ\pi\Big)\end{aligned}$$

We conclude from the previous expression that $\bar{\nabla}$ is the Levi-Civita connection for the spatial metric $\bar{\gamma}\in\Gamma\left(\vee^2\,\mathrm{Ker}\;\bar{\psi}\right)$. □

4.4.3 Koszul connections induced by a Platonic wave

Section 4.4.1 has emphasised several interesting features regarding the embedding of a Koszul connection inside a Bargmann-Eisenhart wave. The first of these features was embodied in Lemma 4.4.1 which asserts that the vector field obtained by parallely transporting (by the means of the Levi-Civita derivative ∇ of the Bargmann-Eisenhart wave) a projectable vector field along another projectable vector field is itself projectable. The proof of this Lemma made crucial use of the fact that ∇ parallelises the wave vector field ξ (the defining characteristic of Bargmann-Eisenhart waves) and, indeed the Lemma does not apply to more general gravitational waves, at least not with the same amount of generality. However, we will show further (*cf.* Lemma 4.4.17) that for the class of Platonic waves, it is possible to mimic Lemma 4.4.1 by restricting the class of projectable vector fields to the one of ξ-invariant vector fields. From now on, we will thus restrict our analysis to ξ-invariant lifts.

The second essential feature encountered in Section 4.4.1 was that the vector field $\bar{\nabla}_{\bar{X}}\bar{Y}\in\Gamma\left(T\bar{\mathscr{M}}\right)$ defined by commutation of Diagram 4.4.18 is independent of the choice of representative lifts $X,Y\in\Gamma\left(T\mathscr{M}\right)$ (*cf.* Lemma 4.4.2). This result is important since it essentially guarantees the uniqueness of the induced derivative operator $\bar{\nabla}$ on the Platonic screen (at least as defined by Diagram 4.4.18). As is turns out, this property is lost when dealing with Platonic waves (even when restricting to ξ-invariant lifts), as will be made manifest in Lemma 4.4.20. Consequently, the definition of a Koszul connection $\bar{\nabla}$ on the Platonic screen from the Levi-Civita connection of a Platonic wave requires an additional prescription in order to fix the arbitrariness in the choice of (ξ-invariant) lifts. This non-uniqueness of $\bar{\nabla}$ in the Platonic case constitutes the main point of discrepancy with Bargmann-Eisenhart waves, since different prescriptions will lead to different Koszul connections. We will propose two examples of prescription and discuss the respective properties of the induced Koszul connections.

Finally, the induced derivative operator $\bar{\nabla}$ has been shown to be a well-defined torsionfree Koszul connection (*cf.* Proposition 4.4.3), before being further characterised as

Galilean (*cf.* Proposition 4.4.7) and then Newtonian (*cf.* Proposition 4.4.9). We commented earlier on the fact that, among Platonic waves, only Bargmann-Eisenhart waves can induce torsionfree Galilean connections so that the various Koszul connections induced by Platonic waves must correspond to different notions of parallelism. We will focus on prescriptions allowing to recover the ones introduced in Section 3.2.3.

Lemma 4.4.17. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave with associated Levi-Civita connection ∇. Let $X, Y \in \Gamma(T\mathscr{M})$ be two ξ-invariant vector fields. Then the vector field $\nabla_X Y \in \Gamma(T\mathscr{M})$ is ξ-invariant.*

Proof: The Killing property enjoyed by the wave vector field $\xi \in \Gamma(T\mathscr{M})$ of a Platonic wave ensures that ξ is ∇-preserving (*cf.* Proposition A.9.10) so that

$$[\xi, \nabla_X Y] = \nabla_{[\xi,X]} Y + \nabla_X [\xi, Y], \ \forall X, Y \in \Gamma(T\mathscr{M}) \tag{4.4.28}$$

(*cf.* Definition A.9.7). Assuming that X and Y are ξ-invariant, the equivariance of $\nabla_X Y$ is straightforward from eq. (4.4.28). $\square$

The proof of Lemma 4.4.17 makes crucial use of the Killing property of the wave vector field ξ (the defining characteristic of Platonic waves) so that this result cannot be naïvely extended to more general classes of gravitational wave. In particular, no similar result seems to be available for Kundt waves. This suggests that Kundt waves, although allowing the embedding of a metric structure (more precisely an Aristotelian structure, *cf.* Proposition 4.2.5), fail to induce a notion of parallelism on the whole Platonic screen.

We now establish the following technical Lemma and its Corollary:

Lemma 4.4.18. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $X \in \Gamma(T\mathscr{M})$ a ξ-invariant vector field. Then the following equality holds*

$$\pi_* (\nabla_X \xi) = \frac{1}{2} \bar{h} \left(d\bar{\psi} \left(\bar{X} \right) \right) \tag{4.4.29}$$

where

- *$\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ is the projection of the vector field X on the Platonic screen ($\bar{X} \equiv \pi_* X$)*
- *$\bar{h} \in \Gamma\left(\vee^2 T\bar{\mathscr{M}}\right)$ is the nonrelativistic contravariant metric on the Platonic screen defined by projection of the contravariant metric $g^{-1} \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ ($\bar{h} \equiv \pi_* g^{-1}$)*
- *$\bar{\psi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ is the nonrelativistic absolute clock defined by projection of the wave covector field $\psi \in \Omega^1(\mathscr{M})$ ($\pi^* \bar{\psi} \equiv \psi$).*

Denoting $\Omega \in C^\infty(\mathscr{M})$ the conformal factor relating the Platonic wave $(\mathscr{M}, g, \xi)$ to a Bargmann-Eisenhart wave, the 2-form $d\psi$ takes the form $d\psi = d \ln \Omega \wedge \psi$, so that expression

(4.4.29) *becomes*

$$\pi_* (\nabla_X \xi) = -\frac{1}{2}\bar{h} (d \ln \Omega)\ \bar{\psi} \left(\bar{X}\right). \tag{4.4.30}$$

Proof: Note first that Lemma 4.4.17 ensures the equivariance of $\nabla_X \xi$ (X and ξ being ξ-invariant) so that the projection $\pi_* (\nabla_X \xi)$ is well-defined. Let $\alpha \in \Omega^1 (\mathscr{M})$ be a projectable 1-form with projection $\bar{\alpha} \in \Omega^1 \left(\bar{\mathscr{M}}\right)$, so that $\pi^* \bar{\alpha} \equiv \alpha$ and denote $Y \in \Gamma (T\mathscr{M})$ the vector field dual to α *i.e.* $Y \equiv g^{-1} (\alpha)$. The vector field Y is ξ-invariant, due to the projectability of α and the Killing property of ξ. Equality (A.9.19) allows to write: $d\psi (X, Y) = g (\nabla_X \xi, Y) - g (\nabla_Y \xi, X)$. The torsionfree condition as well as the equivariance of Y ensures that $\nabla_Y \xi = \nabla_\xi Y$, so that the previous expression can be restated after staightforward manipulations, as:

$$\begin{aligned}
d\psi (X, Y) &= g (\nabla_X \xi, Y) - g (\nabla_\xi Y, X) \\
d\psi \left(X, g^{-1} (\alpha)\right) &= g \left(\nabla_X \xi, g^{-1} (\alpha)\right) - g \left(\nabla_\xi \left(g^{-1} (\alpha)\right), X\right) \\
\alpha \left(g^{-1} (d\psi (X))\right) &= \alpha (\nabla_X \xi) - (\nabla_\xi \alpha) (X).
\end{aligned}$$

where $d\psi (Z) \in \Omega^1 (\mathscr{M})$ stands for $d\psi (Z, \cdot)$ with $Z \in \Gamma (T\mathscr{M})$. Note that the metric compatibility of g^{-1} has been used in order to reformulate the second term on the right-hand side. Now, this term, using eq.(A.9.12), can be expressed as: $(\nabla_\xi \alpha) (X) = \xi [\alpha (X)] - \alpha (\nabla_\xi X)$, where the first term vanishes due to the equivariance of α and X. Plugging back into the previous expression gives:

$$\begin{aligned}
\alpha \left(g^{-1} (d\psi (X))\right) &= \alpha (\nabla_X \xi) + \alpha (\nabla_\xi X) \\
\alpha \left(g^{-1} (d\psi (X))\right) &= 2\alpha (\nabla_X \xi).
\end{aligned}$$

Since all the terms involved admit a well-defined projection on the Platonic screen, the following relation holds: $\bar{\alpha} \left(\bar{h} \left(d\bar{\psi} \left(\bar{X}\right)\right)\right) = 2\bar{\alpha} (\pi_* (\nabla_X \xi))$. Since this relation holds for all $\bar{\alpha} \in \Omega^1 \left(\bar{\mathscr{M}}\right)$, one concludes: $\pi_* (\nabla_X \xi) = \frac{1}{2}\bar{h} \left(d\bar{\psi} \left(\bar{X}\right)\right)$. $\square$

Corollary 4.4.19. *Let $(\mathscr{M}, g, \xi)$ be a proper Platonic wave and $X \in \Gamma (T\mathscr{M})$ a ξ-invariant vector field on $\mathscr{M}$. The vector field $\nabla_X \xi \in \Gamma (T\mathscr{M})$ is colinear with the wave vector field ξ if and only if X is ξ-orthogonal (cf. Notation 4.2.1).*

Proof: The ξ-invariant vector field $\nabla_X \xi$ is colinear with ξ if and only if $\pi_* (\nabla_X \xi) = 0$. From eq.(4.4.30), this implies that the vector field $\bar{h} (d \ln \Omega)\ \bar{\psi} \left(\bar{X}\right)$ must vanish. For a proper Platonic wave (*i.e.* not a Bargmann-Eisenhart wave), this is the case if and only if $\bar{\psi} \left(\bar{X}\right) = 0$, so that $\bar{X}$ is spacelike, hence X is ξ-orthogonal. $\square$

The previous Lemma is instrumental in order to establish the following result, which measures the dependence of $\pi_*(\nabla_X Y)$ on the choice of representative lifts X, Y (compare with Lemma 4.4.2 in the Bargmann-Eisenhart case):

Lemma 4.4.20. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $(\bar{X}, \bar{Y}) \in \Gamma(T\bar{\mathscr{M}}) \times \Gamma(T\bar{\mathscr{M}})$ be a couple of vector fields on the Platonic screen. Moreover, let (X', Y') and $(X, Y) \in \Gamma(T\mathscr{M}) \times \Gamma(T\mathscr{M})$ be two couples of ξ-invariant lifts of $(\bar{X}, \bar{Y})$ related by*

$$\begin{cases} X' = X + f\xi \\ Y' = Y + l\xi \end{cases} \tag{4.4.31}$$

where $f, l \in C^\infty(\mathscr{M})$ are two ξ-invariant functions on $\mathscr{M}$. Then the following equality holds

$$\pi_*(\nabla_{X'} Y') = \pi_*(\nabla_X Y) - \frac{1}{2}\bar{h}(d\ln\Omega)\left(\bar{l}\,\bar{\psi}(\bar{X}) + \bar{f}\,\bar{\psi}(\bar{Y})\right) \tag{4.4.32}$$

where

- *$\bar{X} \in \Gamma(T\bar{\mathscr{M}})$ (resp. $\bar{Y} \in \Gamma(T\bar{\mathscr{M}})$) is the projection of the vector fields X' and X (resp. Y' and Y) on the Platonic screen ($\bar{X} \equiv \pi_* X' = \pi_* X$ and $\bar{Y} \equiv \pi_* Y' = \pi_* Y$, respectively)*
- *$\bar{h} \in \Gamma(\vee^2 T\bar{\mathscr{M}})$ is the nonrelativistic contravariant metric on the Platonic screen defined by projection of the contravariant metric $g^{-1} \in \Gamma(\vee^2 T\mathscr{M})$ ($\bar{h} \equiv \pi_* g^{-1}$)*
- *$\bar{\psi} \in \Omega^1(\bar{\mathscr{M}})$ is the nonrelativistic absolute clock defined by projection of the wave covector field $\psi \in \Omega^1(\mathscr{M})$ ($\pi^*\bar{\psi} \equiv \psi$).*
- *$\bar{f} \in C^\infty(\bar{\mathscr{M}})$ (resp. $\bar{l} \in C^\infty(\bar{\mathscr{M}})$) is the projection of f (resp. l) on the Platonic screen ($f \equiv \bar{f} \circ \pi$ and $l \equiv \bar{l} \circ \pi$, respectively)*

Proof: Note that Proposition 4.1.11 guarantees the existence of two ξ-invariant functions $f, l \in C^\infty(\mathscr{M})$ such that eq.(4.4.31) holds. The vector field $\nabla_{X'} Y' \in \Gamma(T\mathscr{M})$ can then be expanded as:

$$\begin{aligned} \nabla_{X'} Y' &= \nabla_{X+f\xi}(Y + l\xi) \\ &= = \nabla_X Y + X[l]\,\xi + l\nabla_X \xi + f\nabla_\xi Y + f\xi[l]\,\xi + fl\nabla_\xi \xi \end{aligned}$$

where the two last term on the right-hand side are seen to vanish, using the equivariance of l and that ξ is affine geodesic. The fourth term on the right-hand side can be reformulated as $f\nabla_Y \xi$ using the fact that ∇ is torsionfree and the equivariance of Y. Now, projecting on the Platonic screen:

$$\pi_*(\nabla_{X'} Y') = \pi_*(\nabla_X Y) + \bar{l}\,\pi_*(\nabla_X \xi) + \bar{f}\,\pi_*(\nabla_Y \xi)$$

where $\bar{f}, \bar{l} \in C^\infty\left(\bar{\mathscr{M}}\right)$ are the projections of f and l, respectively (*cf.* Proposition 4.1.7). Now, Lemma 4.4.18 provides us with the equality (4.4.30) which can be used to reformulate the previous expression as:

$$\begin{aligned}
\pi_*\left(\nabla_{X'}Y'\right) &= \pi_*\left(\nabla_X Y\right) - \frac{1}{2}\bar{l}\,\bar{h}\,(d\ln\Omega)\;\bar{\psi}\left(\bar{X}\right) - \frac{1}{2}\bar{f}\,\bar{h}\,(d\ln\Omega)\;\bar{\psi}\left(\bar{Y}\right)\\
&= \pi_*\left(\nabla_X Y\right) - \frac{1}{2}\bar{h}\,(d\ln\Omega)\;\left(\bar{l}\,\bar{\psi}\left(\bar{X}\right) + \bar{f}\,\bar{\psi}\left(\bar{Y}\right)\right).
\end{aligned}$$

□

As foreshadowed, Lemma 4.4.20 fails to reproduce Lemma 4.4.2 in the Platonic case, even when restricted to ξ-invariant lifts. In consequence, the projection of the Levi-Civita connection on the Platonic screen depends on the choice of lifts, so that one needs an additional prescription in order to induce a well-defined derivative operator on $\bar{\mathscr{M}}$. We will propose two examples of prescription allowing to induce a well-defined Koszul connection on the Platonic screen of a Platonic wave. The first one will make use of a notion of horizontality on the principal bundle in the guise of an Ehresmann connection $\overset{N}{A}$, allowing to recover the nonrelativistic Horizontal connection defined in Section 3.2.3. The second prescription (*cf.* Section 4.4.5) will make contact with the notion of nonrelativistic Platonic connection and will provide the tools necessary to a geometric understanding of the Eisenhart-Lichnerowicz lift.

4.4.4 Horizontal lift

Proposition 4.4.21. *Let $\mathscr{M}$ be a Platonic wave and $\overset{N}{A} \in \Omega^1(\mathscr{M})$ an Ehresmann connection on $\mathscr{M}$. The derivative operator $\overset{\bar{N}}{\nabla} : \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right) \to \Gamma\left(T\bar{\mathscr{M}}\right)$ defined by commutation of the following diagram*

$$\begin{tikzcd}
(X,Y) \arrow[r, "\nabla"] \arrow[d, "\pi_*"] & \nabla_X Y \arrow[d, "\pi_*"] \\
\left(\bar{X},\bar{Y}\right) \arrow[r, "\overset{\bar{N}}{\nabla}"] & \overset{\bar{N}}{\nabla}_{\bar{X}}\bar{Y}.
\end{tikzcd}$$

where X and $Y \in \Gamma(T\mathscr{M})$ are the horizontal lifts with respect to the Ehresmann connection $\overset{N}{A}$ of the vector fields $\bar{X}$ and $\bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$, respectively, is a well-defined Koszul connection on the Platonic screen $\bar{\mathscr{M}}$.

Proof: First note that Proposition 4.3.13 ensures that X and Y are ξ-invariant, so that $\nabla_X Y$ is ξ-invariant (*cf.* Lemma 4.4.17). The equivariance of $\nabla_X Y$, together with the uniqueness of the horizontal lift, ensures that the projection $\pi_*(\nabla_X Y)$ is well-defined. That the derivative operator $\overset{\bar{N}}{\nabla}$ thus defined satisfies the Axioms of a Koszul connection (*cf.* Definition A.9.1) can be proved mirroring the steps followed in the proof of Proposition 4.4.3. $\square$

The nonrelativistic Koszul connection $\overset{\bar{N}}{\nabla}$ will soon be given the interpretation of a Horizontal connection on the Platonic screen $\bar{\mathscr{M}}$. Before that, we prove the two following technical Lemmas:

Lemma 4.4.22. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $\overset{N}{A} \in \Omega^1(\mathscr{M})$ an Ehresmann connection on $\mathscr{M}$. The Koszul connection defined in Proposition 4.4.21 acts on 1-forms as:*

$$\left(\overset{\bar{N}}{\nabla}_{\bar{X}}\bar{\alpha}\right)\left(\bar{Y}\right)\circ\pi = \left(\nabla_X \alpha\right)(Y) \tag{4.4.33}$$

with

- $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ *a 1-form on the Platonic screen*
- $\alpha \in \Omega^1(\mathscr{M})$ *the pullback of $\bar{\alpha}$ by the projection π*
- $\bar{X}, \bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ *two vector fields on the Platonic screen*
- $X, Y \in \Gamma(T\mathscr{M})$ *the respective horizontal lifts of $\bar{X}$ and $\bar{Y}$ with respect to $\overset{N}{A}$.*

Proof: Equality (4.4.33) is straightforwardly established starting from expression (A.9.12) for the Levi-Civita connection associated to g and manipulating using formulas of Section A.2:

$$\begin{aligned}
(\nabla_X\alpha)(Y) &= X[\alpha(Y)] - \alpha(\nabla_X Y) \\
&= X[\pi^*\bar{\alpha}(Y)] - \pi^*\bar{\alpha}(\nabla_X Y) \\
&= X\left[\bar{\alpha}\left(\bar{Y}\right)\circ\pi\right] - \bar{\alpha}\left(\pi_*(\nabla_X Y)\right)\circ\pi \\
&= \bar{X}\left[\bar{\alpha}\left(\bar{Y}\right)\right]\circ\pi - \bar{\alpha}\left(\overset{\bar{N}}{\nabla}_{\bar{X}}\bar{Y}\right)\circ\pi \\
&= \left(\overset{\bar{N}}{\nabla}_{\bar{X}}\bar{\alpha}\right)\left(\bar{Y}\right)\circ\pi.
\end{aligned}$$

$\square$

Lemma 4.4.23. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $\overset{N}{A} \in \Omega^1(\mathscr{M})$ an Ehresmann connection on $\mathscr{M}$. Furthermore, let $\bar{\alpha} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ be a 1-form on the Platonic screen $\bar{\mathscr{M}}$. The horizontal lift of the vector field $\bar{h}(\bar{\alpha}) \in \Gamma\left(T\bar{\mathscr{M}}\right)$ is given by $\overset{N}{h}(\alpha) \in \Gamma(T\mathscr{M})$, with*

$\alpha \in \Omega^1(\mathscr{M})$ the lift of $\bar{\alpha}$ on $\mathscr{M}$ and where the contravariant metric $\overset{N}{h} \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ is defined as $\overset{N}{h} \equiv g^{-1} - N \otimes \xi - \xi \otimes N$, with $N \equiv g^{-1}\left(\overset{N}{A}\right)$.

Proof: We first prove that $\overset{N}{h}(\alpha)$ is ξ-invariant. This is easily seen by writing

$$\mathcal{L}_\xi\left(\overset{N}{h}(\alpha)\right) = \left(\mathcal{L}_\xi \overset{N}{h}\right)(\alpha) + \overset{N}{h}(\mathcal{L}_\xi \alpha) = 0$$

where the respective equivariance of g, $\overset{N}{A}$, N and ξ has been used. The projection of $\overset{N}{h}(\alpha)$ is then well-defined and we have $\pi_*\left(\overset{N}{h}(\alpha)\right) = \bar{h}(\bar{\alpha})$. The horizontality of $\overset{N}{h}(\alpha)$ is now shown as follows:

$$\begin{aligned}\overset{N}{A}\left(\overset{N}{h}(\alpha)\right) &= \overset{N}{h}\left(\overset{N}{A}, \alpha\right) = g^{-1}\left(\overset{N}{A}, \alpha\right) - \overset{N}{A}(N)\,\alpha(\xi) - \overset{N}{A}(\xi)\,\alpha(N) \\ &= \alpha(N) - \alpha(N) = 0\end{aligned}$$

where we used that N and ξ are respectively a horizontal and a fundamental vector field. $\square$

Proposition 4.4.24. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $\overset{N}{A} \in \Omega^1(\mathscr{M})$ an Ehresmann connection on $\mathscr{M}$. The Koszul connection $\overset{\bar{N}}{\nabla}$ defined in Proposition 4.4.21 is a Horizontal connection. (cf. Definition 3.2.46).*

Proof: Let $\bar{X}, \bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ be two vector fields on the Platonic screen $\bar{\mathscr{M}}$ and designate $X, Y \in \Gamma(T\mathscr{M})$ their respective horizontal lifts with respect to the Ehresmann connection $\overset{N}{A}$. We start by showing that $\overset{\bar{N}}{\nabla}$ is torsion-free and then that the Axioms 1-3 of Proposition 3.2.45 are satisfied:

- The torsionfree condition satisfied by the Levi-Civita connection associated to the Platonic metric g allows to write: $\nabla_X Y - \nabla_Y X - [X, Y] = 0$. Projecting on $\bar{\mathscr{M}}$, one obtains the relation $\pi_*(\nabla_X Y) - \pi_*(\nabla_Y X) - \pi_*[X, Y] = 0$. Using Definition 3.2.46 and Proposition 4.1.12 leads to

$$\overset{\bar{N}}{\nabla}_{\bar{X}} \bar{Y} - \overset{\bar{N}}{\nabla}_{\bar{Y}} \bar{X} - \left[\bar{X}, \bar{Y}\right] = 0.$$

- Axiom 1: Platonic waves are characterised by the existence of a Killing wave vector field so that one is allowed to write (*cf.* Proposition A.9.9):

$$g\left(\nabla_X \xi, Y\right) + g\left(X, \nabla_Y \xi\right) = 0.$$

The compatibility of ∇ with g ensures that $\left(\nabla_X \psi\right)(Y) + \left(\nabla_Y \psi\right)(X) = 0$. Projecting on the Platonic screen and using Lemma 4.4.22 leads to

$$\left(\overset{\bar{N}}{\nabla}_{\bar{X}} \bar{\psi}\right)\left(\bar{Y}\right) + \left(\overset{\bar{N}}{\nabla}_{\bar{Y}} \bar{\psi}\right)\left(\bar{X}\right) = 0$$

so that the absolute clock $\bar{\psi} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ is a conservor (*cf.* Definition 3.2.42).

– Axiom 2: We start from the compatibility condition of the Platonic Levi-Civita connection with the contravariant metric g^{-1} expressed as:

$$X\left[g^{-1}(\alpha, \beta)\right] = g^{-1}\left(\nabla_X \alpha, \beta\right) + g^{-1}\left(\alpha, \nabla_X \beta\right) = 0$$

where $\alpha, \beta \in \Omega^1(\mathscr{M})$ are chosen to be two projectable 1-forms on $\mathscr{M}$ with respective projections $\bar{\alpha}, \bar{\beta} \in \Omega^1\left(\bar{\mathscr{M}}\right)$. Replacing g^{-1} by $g^{-1} = \overset{N}{h} + 2N \vee \xi$, one obtains:

$$\begin{aligned} X\left[\overset{N}{h}(\alpha, \beta)\right] &= \overset{N}{h}\left(\nabla_X \alpha, \beta\right) + \overset{N}{h}\left(\alpha, \nabla_X \beta\right) \\ &\quad + \left(\nabla_X \alpha\right)(\xi)\, \beta(N) + \left(\nabla_X \beta\right)(\xi)\, \alpha(N) \end{aligned} \tag{4.4.34}$$

where one used that $\alpha(\xi) = \beta(\xi) = 0$. We now make the additional hypothesis that $\bar{\alpha}, \bar{\beta} \in \operatorname{Ann} \bar{N}$. Consequently, $\alpha(N) = \beta(N) = 0$, so that

$$X\left[\overset{N}{h}(\alpha, \beta)\right] = \overset{N}{h}\left(\nabla_X \alpha, \beta\right) + \overset{N}{h}\left(\alpha, \nabla_X \beta\right).$$

Making use of Lemmas 4.4.22 and 4.4.23 allows to write

$$\overset{N}{h}\left(\nabla_X \alpha, \beta\right) = \left(\nabla_X \alpha\right)\left(\overset{N}{h}(\beta)\right) = \left(\overset{\bar{N}}{\nabla}_{\bar{X}} \bar{\alpha}\right)\left(\bar{h}\left(\bar{\beta}\right)\right) \circ \pi = \bar{h}\left(\overset{\bar{N}}{\nabla}_{\bar{X}} \bar{\alpha}, \bar{\beta}\right) \circ \pi.$$

so that expression (4.4.34) projects as:

$$\bar{X}\left[\bar{h}\left(\bar{\alpha}, \bar{\beta}\right)\right] = \bar{h}\left(\overset{\bar{N}}{\nabla}_{\bar{X}} \bar{\alpha}, \bar{\beta}\right) + \bar{h}\left(\bar{\alpha}, \overset{\bar{N}}{\nabla}_{\bar{X}} \bar{\beta}\right)$$

which ensures that Axiom 2 of Proposition 3.2.45 is satisfied.

– Axiom 3: We start again from eq.(4.4.34) and make the hypothesis that X is

ξ-orthogonal. Now, Proposition 6 of Lemma 4.4.12 ensures that $(\nabla_X \alpha)(\xi) = (\nabla_X \beta)(\xi) = 0$. Following the same steps as in the proof of Axiom 2 guarantees that Axiom 3 of Proposition 3.2.45 holds.

□

4.4.5 Orthogonal lift

Definition 4.4.25 (Orthogonal lift). *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $\left(\bar{X}, \bar{Y}\right) \in \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right)$ be a couple of vector fields on the Platonic screen $\bar{\mathscr{M}}$ such that $\bar{X}$ and $\bar{Y}$ are not both spacelike. The couple of vector fields $(X, Y) \in \Gamma(T\mathscr{M}) \times \Gamma(T\mathscr{M})$ on $\mathscr{M}$ is said to be an orthogonal lift of (X, Y) if X and Y satisfy the conditions:*

- *X and Y are ξ-invariant*
- *$\pi_* X = \bar{X}$ and $\pi_* Y = \bar{Y}$*
- *$g(X, Y) = 0$.*

Lemma 4.4.26. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave . To any couple $\left(\bar{X}, \bar{Y}\right) \in \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right)$ of vector fields on the Platonic screen $\bar{\mathscr{M}}$ such that $\bar{X}$ and $\bar{Y}$ are not both spacelike corresponds a class $[(X, Y)]$ of orthogonal lifts of (X, Y).*

Proof: Let $X', X \in \Gamma(T\mathscr{M})$ (resp. $Y', Y \in \Gamma(T\mathscr{M})$) be two ξ-invariant lifts of $\bar{X}$ (resp. of $\bar{Y}$). By Proposition 4.1.11, there exists a ξ-invariant function $f \in C^\infty(\mathscr{M})$ (resp. $l \in C^\infty(\mathscr{M})$) such that $X' = X + f\xi$ (resp. $Y' = Y + l\xi$). Imposing $g(X', Y') = 0$ yields $g(X, Y) + l\psi(X) + f\psi(Y) = 0$. By hypothesis, $\bar{X}$ and $\bar{Y}$ are not both spacelike. Assuming that $\bar{X}$ is not spacelike, then $\psi(X) \neq 0$ and the previous equation can always be solved for l. □

According to the proof of Lemma 4.4.26, if the couples (X', Y') and (X, Y) belong to the class of orthogonal lifts for the couple $\left(\bar{X}, \bar{Y}\right)$, then there exist two ξ-invariant functions $f, l \in C^\infty(\mathscr{M})$ such that:

$$\begin{cases} X' = X + f\xi \\ Y' = Y + l\xi \\ l\psi(X) + f\psi(Y) = 0. \end{cases} \tag{4.4.35}$$

Lemma 4.4.27. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $\left(\bar{X}, \bar{Y}\right) \in \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right)$ be a couple of vector fields on the Platonic screen $\bar{\mathscr{M}}$ such that $\bar{X}$ and $\bar{Y}$ are not both spacelike. Let (X', Y') and (X, Y) be two orthogonal lifts for the couple $\left(\bar{X}, \bar{Y}\right)$. Then*

$$\pi_*\left(\nabla_{X'} Y'\right) = \pi_*\left(\nabla_X Y\right). \tag{4.4.36}$$

Proof: The proof is obtained straightforwardly by inserting the (projection of the) third equality of (4.4.35) into eq.(4.4.32). □

As hinted in the proof of Lemma 4.4.27, imposing the lifts of two nonrelativistic vector fields to be orthogonal is the most minimal prescription ensuring that the Levi-Civita parallel transport projects well (*i.e.* independently of the remaining arbitrariness), since eq.(4.4.35) provides just the restriction needed in order for eq.(4.4.36) to hold. The notion of orthogonal lift is natural in this respect as it leaves room for a certain amount of freedom, as embodied in eq.(4.4.35), in contradistinction *e.g.* , with the horizontal prescription which fixed uniquely the choice of lifts.

In order to be exhaustive, one needs an additional prescription concerning the lifts of two spacelike vector fields $\left(\bar{X},\bar{Y}\right)$, since the orthogonal lift is inapplicable in this specific case. A natural option consists in taking advantage of the affine character of the space of timelike vector fields. The spacelike vector field $\bar{X}$ is thus envisaged as a sum of two timelike vector fields $\bar{X}=\bar{X}_1+\bar{X}_2$, with $\bar{X}_1,\bar{X}_2\in\Gamma\left(T\bar{\mathscr{M}}\right)$, so that the orthogonal lifts of $\left(\bar{X}_1,\bar{Y}\right)$ and $\left(\bar{X}_2,\bar{Y}\right)$ are well-defined. The following Definition and Lemma provide a more precise formalisation of this line of reasoning:

Definition 4.4.28 (Orthogonal operator)**.** *Let $(\mathscr{M},g,\xi)$ be a Platonic wave with Levi-Civita connection ∇ and Platonic screen $\bar{\mathscr{M}}$. Let $\bar{X}$ and $\bar{Y}\in\Gamma\left(T\bar{\mathscr{M}}\right)$ be two vector fields on $\bar{\mathscr{M}}$. The orthogonal operator $\bar{\nabla}:\Gamma\left(T\bar{\mathscr{M}}\right)\times\Gamma\left(T\bar{\mathscr{M}}\right)\to\Gamma\left(T\bar{\mathscr{M}}\right)$ is defined as:*

- *$\bar{X}$ and $\bar{Y}$ are not both spacelike: $\bar{\nabla}_{\bar{X}}\bar{Y}\equiv\pi_*\left(\nabla_XY\right)$, with $(X,Y)\in\Gamma\left(T\mathscr{M}\right)\times\Gamma\left(T\mathscr{M}\right)$ an orthogonal lift of $\left(\bar{X},\bar{Y}\right)$*
- *$\bar{X}$ and $\bar{Y}$ are both spacelike: $\bar{\nabla}_{\bar{X}}\bar{Y}\equiv\bar{\nabla}_{\bar{X}_1}\bar{Y}+\nabla_{\bar{X}_2}\bar{Y}$ where*

$$\begin{cases}\bar{X}_1=\frac{1}{2}\left(\bar{X}+\bar{N}\right)\\ \bar{X}_2=\frac{1}{2}\left(\bar{X}-\bar{N}\right)\end{cases}\tag{4.4.37}$$

with $\bar{N}\in\Gamma\left(T\bar{\mathscr{M}}\right)$ an arbitrary timelike vector field.

Since $\bar{X}_1$ and $\bar{X}_2$ are timelike, the definition of $\bar{\nabla}_{\bar{X}_1}\bar{Y}$ (resp. $\bar{\nabla}_{\bar{X}_2}\bar{Y}$) falls within the ambit of the first item, and is then computed by taking the orthogonal lift of $\left(\bar{X}_1,\bar{Y}\right)$, (resp. $\left(\bar{X}_2,\bar{Y}\right)$).

Lemma 4.4.29. *Let $(\mathscr{M},g,\xi)$ be a Platonic wave and $\left(\bar{X},\bar{Y}\right)\in\Gamma\left(T\bar{\mathscr{M}}\right)\times\Gamma\left(T\bar{\mathscr{M}}\right)$ be a couple of spacelike vector fields on the Platonic screen $\bar{\mathscr{M}}$. The Platonic screen is endowed with the orthogonal operator (cf. Definition 4.4.28) $\bar{\nabla}:\Gamma\left(T\bar{\mathscr{M}}\right)\times\Gamma\left(T\bar{\mathscr{M}}\right)\to\Gamma\left(T\bar{\mathscr{M}}\right)$. The following relation holds:*

$$\bar{\nabla}_{\bar{X}}\bar{Y}=\pi_*\left(\nabla_XY\right)+\frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{Y}\right)\tag{4.4.38}$$

with

- $X \in \Gamma(T\mathscr{M})$ *an arbitrary lift of* $\bar{X}$
- $Y \in \Gamma(T\mathscr{M})$ *an arbitrary lift of* $\bar{Y}$
- $\Omega \in C^\infty(\mathscr{M})$ *the conformal factor relating the Platonic wave* $(\mathscr{M}, g, \xi)$ *to a Bargmann-Eisenhart wave*
- $\bar{\gamma} \in \Gamma\left(\vee^2 \operatorname{Ker} \bar{\psi}\right)$ *the covariant spatial metric on the absolute spaces*
- $\bar{h} \in \Gamma\left(\vee^2 T\bar{\mathscr{M}}\right)$ *the contravariant metric on the Platonic screen.*

Proof: According to Definition 4.4.28, since $\bar{X}$ and $\bar{Y}$ are both spacelike, one needs to introduce the vector fields

$$\begin{cases} \bar{X}_1 = \frac{1}{2}\left(\bar{X} + \bar{N}\right) \\ \bar{X}_2 = \frac{1}{2}\left(\bar{X} - \bar{N}\right) \end{cases}$$

where $\bar{N} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ is a timelike vector field on $\bar{\mathscr{M}}$.
Let us denote (X_1, Y_1) an orthogonal lift for the couple $\left(\bar{X}_1, \bar{Y}\right)$. The orthogonality condition $g(X_1, Y_1) = 0$ fixes uniquely the lift $Y_1 \in \Gamma(T\mathscr{M})$, since $\bar{Y}$ is spacelike. Proceeding similarly and denoting (X_2, Y_2) an orthogonal lift for the couple $\left(\bar{X}_2, \bar{Y}\right)$, the orthogonality condition $g(X_2, Y_2) = 0$ determines Y_2 uniquely. A priori, $Y_1 \neq Y_2$ but these two lifts of $\bar{Y}$ must satisfy $Y_2 = Y_1 + f\xi$ for some ξ-invariant function $f \in C^\infty(\mathscr{M})$. Denoting $X \equiv X_1 + X_2$ and using both orthogonality condition allows to write:

$$\begin{aligned} g(X_2, Y_2) &= g(X_2, Y_1 + f\xi) \\ &= g(X_2, Y_1) + f\psi(X_2) \\ &= g(X, Y_1) - g(X_1, Y_1) + f\psi(X_2), \end{aligned}$$

so that $g(X, Y_1) + f\psi(X_2) = 0$, which projects as

$$\bar{\gamma}\left(\bar{X}, \bar{Y}\right) + \bar{f}\bar{\psi}\left(\bar{X}_2\right) = 0, \tag{4.4.39}$$

where $\bar{f}$ denotes the projection of f on the Platonic screen.

Now, the definition of $\bar{\nabla}_{\bar{X}}\bar{Y}$ can be expressed as :

$$\begin{aligned}
\bar{\nabla}_{\bar{X}}\bar{Y} &= \bar{\nabla}_{\bar{X}_1}\bar{Y} + \nabla_{\bar{X}_2}\bar{Y}\\
&= \pi_*\left(\nabla_{X_1}Y_1 + \nabla_{X_2}Y_2\right)\\
&= \pi_*\left(\nabla_{X_1}Y_1 + \nabla_{X_2}\left(Y_1 + f\xi\right)\right)\\
&= \pi_*\left(\nabla_{X_1}Y_1 + \nabla_{X_2}Y_1 + X_2\left[f\right]\xi + f\nabla_{X_2}\xi\right)\\
&= \pi_*\left(\nabla_X Y_1\right) + \pi_*\left(f\nabla_{X_2}\xi\right).
\end{aligned}$$

Using eq.(4.4.30) allows to reformulate the second term on the right-hand side as $\pi_*\left(f\nabla_{X_2}\xi\right) = -\frac{1}{2}\,\bar{f}\,\bar{h}\left(d\ln\Omega\right)\bar{\psi}\left(\bar{X}_2\right) = \frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{Y}\right)$, where eq.(4.4.39) has been used. Note that the term $\pi_*\left(\nabla_X Y_1\right)$ does not depend on the choice of lifts X and Y_1, in virtue of Lemma 4.4.14. □

Two main conclusions can be drawn from Lemma 4.4.29:

- For $\bar{X}$ and $\bar{Y}$ both spacelike, $\bar{\nabla}_{\bar{X}}\bar{Y}$ is independent of the choice of $\bar{N}$, with $\bar{\nabla}$ the orthogonal operator and $\bar{N} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ the timelike vector field appearing in the second item of Definition 4.4.28. This ensures that the definition of the orthogonal operator is consistent for spacelike vector fields.
- Proposition 4.4.16 established that the Levi-Civita connection of Kundt waves projects well on the absolute spaces of their associated Platonic screen as the Levi-Civita connection for the spatial metric $\bar{\gamma}$. In view of eq.(4.4.29), one concludes that the orthogonal operator does not generically agree with the spatial Levi-Civita connection on absolute spaces.

Proposition 4.4.30. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave with Platonic screen $\bar{\mathscr{M}}$. The orthogonal operator $\bar{\nabla} : \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right) \to \Gamma\left(T\bar{\mathscr{M}}\right)$ is a Koszul connection.*

Proof: We first prove the linearity axioms

$$\bar{\nabla}_{\bar{X}}\left(\bar{Y} + \bar{Z}\right) = \bar{\nabla}_{\bar{X}}\bar{Y} + \bar{\nabla}_{\bar{X}}\bar{Z} \tag{4.4.40}$$
$$\bar{\nabla}_{\bar{Y}+\bar{Z}}\bar{X} = \bar{\nabla}_{\bar{Y}}\bar{X} + \bar{\nabla}_{\bar{Z}}\bar{X}. \tag{4.4.41}$$

by making use of the two following Lemmas, which distinguish according to whether the vector field $\bar{X}$ is timelike or spacelike:

Lemma 4.4.31. *Let $\bar{X}, \bar{Y}$ and $\bar{Z} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ three vector fields on $\bar{\mathscr{M}}$ such that $\bar{X}$ is timelike. Then the linearity conditions* (4.4.40)-(4.4.41) *hold.*

Proof: We start from the linearity of the Levi-Civita connection ∇ of a Platonic wave: $\nabla_X\left(Y + Z\right) = \nabla_X Y + \nabla_X Z$ where $X \in \Gamma\left(T\mathscr{M}\right)$ is assumed to be a lift of $\bar{X}$. Now, imposing (X, Y) to be an orthogonal lift for the couple $\left(\bar{X}, \bar{Y}\right)$ fixes the lift Y in terms of X, via the orthogonality condition $g(X, Y) = 0$. Similarly, (X, Z) is assumed to be an orthogonal lift for $\left(\bar{X}, \bar{Z}\right)$ which determines uniquely the lift Z in terms of X. By linearity of the metric, the couple $(X, Y + Z)$ is automatically an orthogonal lift for $\left(\bar{X}, \bar{Y} + \bar{Z}\right)$. Consequently, projecting $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$ leads straightforwardly to the linearity condition (4.4.40). The linearity condition (4.4.41) can be proved along similar lines. $\square$

Lemma 4.4.32. *Let $\bar{X}, \bar{Y}$ and $\bar{Z} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ three vector fields on $\bar{\mathscr{M}}$ such that $\bar{X}$ is spacelike. Then the linearity conditions* (4.4.40)-(4.4.41) *hold.*

Proof: Since $\bar{X}$ is assumed to be spacelike, it can be written as $\bar{X} \equiv \bar{X}_1 + \bar{X}_2$ where $\bar{X}_1, \bar{X}_2 \in \Gamma\left(T\bar{\mathscr{M}}\right)$ are timelike vector fields on $\bar{\mathscr{M}}$ defined as:

$$\begin{cases} \bar{X}_1 = \frac{1}{2}\left(\bar{X} + \bar{N}\right) \\ \bar{X}_2 = \frac{1}{2}\left(\bar{X} - \bar{N}\right) \end{cases}$$

with $\bar{N} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ an arbitrary timelike vector field. This decomposition allows to write $\bar{\nabla}_{\bar{X}}\left(\bar{Y} + \bar{Z}\right) = \bar{\nabla}_{\bar{X}_1}\left(\bar{Y} + \bar{Z}\right) + \bar{\nabla}_{\bar{X}_2}\left(\bar{Y} + \bar{Z}\right)$. Whenever $\bar{Y} + \bar{Z}$ is timelike, the previous equality follows from eq.(4.4.41) when $\bar{X}$ is timelike, as proved in Lemma 4.4.31. However, when $\bar{X}$ is timelike, the previous equality is just the definition of the orthogonal operator acting on two spacelike vector fields. Since $\bar{X}_1$ and $\bar{X}_2$ are both timelike, eq.(4.4.40) of Lemma 4.4.31 can be used in order distribute the previous relation as: $\bar{\nabla}_{\bar{X}}\left(\bar{Y} + \bar{Z}\right) = \bar{\nabla}_{\bar{X}_1}\bar{Y} + \bar{\nabla}_{\bar{X}_1}\bar{Z} + \bar{\nabla}_{\bar{X}_2}\bar{Y} + \bar{\nabla}_{\bar{X}_2}\bar{Z}$. Now, one can factorise as $\bar{\nabla}_{\bar{X}}\left(\bar{Y} + \bar{Z}\right) = \bar{\nabla}_{\bar{X}_1+\bar{X}_2}\bar{Y} + \bar{\nabla}_{\bar{X}_1+\bar{X}_2}\bar{Z}$, using either eq.(4.4.41) of Lemma 4.4.31 when $\bar{Y}$ (resp. $\bar{Z}$) is timelike or the definition of $\bar{\nabla}_{\bar{X}}\bar{Y}$ (resp. $\bar{\nabla}_{\bar{X}}\bar{Z}$) whenever $\bar{Y}$ (resp. $\bar{Z}$) is spacelike. By definition of $\bar{X}$, this amounts to eq.(4.4.40) when $\bar{X}$ is spacelike. The proof of eq.(4.4.41) in the spacelike case can be performed similarly. $\square$

Regarding, the third and fourth axioms, we first consider the case when $\bar{X}$ and $\bar{Y}$ are not both spacelike. One first notices that (X, Y) is an orthogonal lift for the couple $\left(\bar{X}, \bar{Y}\right)$ if and only if (fX, Y) is an orthogonal lift for $\left(\bar{f}\bar{X}, \bar{Y}\right)$, with $\bar{f} \in C^\infty\left(\bar{\mathscr{M}}\right)$, $\bar{X}, \bar{Y} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ and $f \in C^\infty\left(\mathscr{M}\right)$ the lift of $\bar{f}$. This property is straightforwardly seen from Lemma 4.1.13 and the bilinearity of g. Regarding the third axiom, we start from $\bar{\nabla}_{\bar{f}\bar{X}}\bar{Y} = \pi_*\left(\nabla_{fX}Y\right)$, with (X, Y) an orthogonal lift for the couple $\left(\bar{X}, \bar{Y}\right)$. Using that the Levi-Civita connection ∇ satisfy the third axiom, one obtains: $\bar{\nabla}_{\bar{f}\bar{X}}\bar{Y} = \pi_*\left(f\nabla_X Y\right) = \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}$, where we used that (X, Y) is an orthogonal lift for $\left(\bar{X},\bar{Y}\right)$. The proof of the fouth axiom is equally straightforward:

$$\begin{aligned}
\bar{\nabla}_{\bar{X}}\bar{f}\bar{Y} &= \pi_*\left(\nabla_X fY\right)\\
&= \pi_*\left(X\left[f\right]Y + f\nabla_X Y\right)\\
&= \bar{X}\left[\bar{f}\right]\bar{Y} + \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}.
\end{aligned}$$

Lastly, when $\bar{X}$ and $\bar{Y}$ are both spacelike, Lemma 4.4.29 ensures that $\bar{\nabla}_{\bar{X}}\bar{Y} = \pi_*\left(\nabla_X Y\right) + \frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{Y}\right)$, so that:

$$\begin{aligned}
\bar{\nabla}_{\bar{f}\bar{X}}\bar{Y} &= \pi_*\left(\nabla_{fX}Y\right) + \frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{f}\bar{X},\bar{Y}\right)\\
&= \pi_*\left(f\nabla_X Y\right) + \frac{1}{2}\bar{f}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{Y}\right)\\
&= \bar{f}\left[\pi_*\left(\nabla_X Y\right) + \frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{Y}\right)\right]\\
&= \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\nabla}_{\bar{X}}\left(\bar{f}\bar{Y}\right) &= \pi_*\left(\nabla_X\left(fY\right)\right) + \frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{f}\bar{Y}\right)\\
&= \pi_*\left(X\left[f\right]Y + f\nabla_X Y\right) + \frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{f}\bar{Y}\right)\\
&= \bar{X}\left[\bar{f}\right]\bar{Y} + \bar{f}\left[\pi_*\left(\nabla_X Y\right) + \frac{1}{2}\bar{h}\left(d\ln\Omega\right)\bar{\gamma}\left(\bar{X},\bar{Y}\right)\right]\\
&= \bar{X}\left[\bar{f}\right]\bar{Y} + \bar{f}\bar{\nabla}_{\bar{X}}\bar{Y}.
\end{aligned}$$

□

Proposition 4.4.33. *Let $(\mathscr{M},g,\xi)$ be a Platonic wave with Platonic screen $\bar{\mathscr{M}}$. The orthogonal connection $\bar{\nabla}$ is a Platonic connection for the Aristotelian structure $\mathscr{A}\left(\bar{\mathscr{M}},\bar{\psi},\bar{h}\right)$ induced by $(\mathscr{M},g,\xi)$. If we denote $(\mathscr{M},\hat{g},\xi)$ the Bargmann-Eisenhart wave conformally related to the Platonic wave $(\mathscr{M},g,\xi)$ and $\bar{\hat{\nabla}}$ the Newtonian connection induced by $\hat{g}$ on $\bar{\mathscr{M}}$, then $\bar{\nabla}$ and $\bar{\hat{\nabla}}$ are conformally related.*

Proof: First we note that the Aristotelian structure $\mathscr{A}\left(\bar{\mathscr{M}},\bar{\psi},\bar{h}\right)$ induced by $(\mathscr{M},g,\xi)$ must be conformally related (in the sense of Definition 3.2.50) to the Augustinian structure $\mathscr{S}\left(\mathscr{M},\bar{\hat{\psi}},\bar{\hat{h}}\right)$. Let $\bar{\hat{\nabla}}$ denote the Newtonian connection obtained by projection on the Platonic screen $\bar{\mathscr{M}}$ of the Levi-Civita connection $\hat{\nabla}$ associated to the Bargmann-Eisenhart wave $(\mathscr{M},\hat{g},\xi)$, so that $\mathscr{N}\left(\mathscr{M},\bar{\hat{\psi}},\bar{\hat{h}},\bar{\hat{\nabla}}\right)$ is a Newtonian manifold. The orthogonal Koszul connection on $\bar{\mathscr{M}}$ originating from ∇ will be denoted $\bar{\nabla}$.

Now, denoting $\Omega \in C^\infty(\mathscr{M})$ the conformal factor relating g and $\hat{g}$ (*i.e.* $g = \Omega\,\hat{g}$), the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ and $\hat{\Gamma}^\lambda_{\mu\nu}$ associated to g and $\hat{g}$, respectively are related according to relation (3.1.2) as

$$\Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{2}\left(\delta^\lambda_\mu \partial_\nu \ln\Omega + \delta^\lambda_\nu \partial_\mu \ln\Omega - g^{\lambda\rho} g_{\mu\nu}\,\partial_\rho \ln\Omega\right). \tag{4.4.42}$$

Letting $X, \hat{X} \in \Gamma(T\mathscr{M})$ be two ξ-invariant vector fields on $\mathscr{M}$ satisfying $X = \Omega^{-1}\hat{X}$, their respective projections $\bar{X}$ and $\bar{\hat{X}}$ on $\bar{\mathscr{M}}$ satisfy the relation $\bar{X} = \bar{\Omega}^{-1}\bar{\hat{X}}$, where $\bar{\Omega} \in C^\infty\left(\bar{\mathscr{M}}\right)$ is the projection of the (ξ-invariant by definition) function Ω. Equation (4.4.42) allows to compute the relation

$$\nabla_X X = \Omega^{-2}\hat{\nabla}_{\hat{X}}\hat{X} - \frac{1}{2} g^{-1}(d\ln\Omega)\, g(X, X).$$

Projecting onto $\bar{\mathscr{M}}$, one obtains

$$\pi_*(\nabla_X X) = \bar{\Omega}^{-2}\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}} - \frac{1}{2}\bar{h}\left(d\ln\bar{\Omega}\right) g(X, X). \tag{4.4.43}$$

where the term $g(X, X)$ is intended here as the projection of the ξ-invariant function $g(X, X)$ on $\bar{\mathscr{M}}$. In order to make contact with the orthogonal Koszul connection $\bar{\nabla}$, we need to discriminate between the two following cases:

- X is ξ-orthogonal:

  Since X is ξ-orthogonal, the vector field $\bar{X}$ is spacelike. Definition 4.2.3 and Proposition 4.2.4 thus ensure that $g(X, X)$ projects onto $\bar{\gamma}\left(\bar{X}, \bar{X}\right)$, so that

  $$\pi_*(\nabla_X X) = \bar{\Omega}^{-2}\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}} - \frac{1}{2}\bar{h}\left(d\ln\bar{\Omega}\right)\bar{\gamma}\left(\bar{X}, \bar{X}\right).$$

  On the other hand, Lemma 4.4.29 ensures that

  $$\pi_*(\nabla_X X) = \bar{\nabla}_{\bar{X}}\bar{X} - \frac{1}{2}\bar{h}(d\ln\Omega)\,\bar{\gamma}\left(\bar{X}, \bar{X}\right)$$

  Substracting these last two relations leads to

  $$\bar{\nabla}_{\bar{X}}\bar{X} = \bar{\Omega}^{-2}\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}}. \tag{4.4.44}$$

- X is not ξ-orthogonal:

  Since X is not ξ-orthogonal, it can be chosen to be null with respect to the metric g, *i.e.* $g(X, X) = 0$, so that the last term of eq.(4.4.43) vanishes. Now, since X is null, the couple (X, X) is an orthogonal lift for the timelike vector field $\bar{X}$ and eq.(4.4.43) becomes

$$\bar{\nabla}_{\bar{X}}\bar{X} = \bar{\Omega}^{-2}\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}}. \tag{4.4.45}$$

Equations (4.4.44) and (4.4.45) together ensure that the following assertion holds

$$\forall \bar{X},\, \bar{\hat{X}} \in \Gamma\left(T\bar{\mathscr{M}}\right) \ / \ \bar{X} = \bar{\Omega}^{-1}\bar{\hat{X}}, \quad \bar{\nabla}_{\bar{X}}\bar{X} = \bar{\Omega}^{-2}\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}}.$$

According to Proposition 3.2.52, $\bar{\nabla}$ is then a Platonic connection conformally related to the Newtonian connection $\bar{\hat{\nabla}}$.
□

Generalising the Orthogonal lift

We continue our investigation of orthogonal lifts by discussing an extension thereof that will be proved useful in order to grasp the geometric origin of the scalar potential shift occuring in the Eisenhart-Lichnerowicz lift for relativistic non-null geodesics. Considering two vector fields $\bar{X}$ and $\bar{Y}$ on the Platonic screen $\bar{\mathscr{M}}$ of a Platonic wave $(\mathscr{M}, g, \xi)$, this extension is obtained by substituting in Definition 4.4.25 the orthogonal condition $g\,(X, Y) = 0$ with

$$g\,(X, Y) = -M^2\psi\,(X)\,\psi\,(Y) \tag{4.4.46}$$

where ψ is the wave covector field on $\mathscr{M}$ and M a fixed constant. We are then led to define:

Definition 4.4.34 (Generalised Orthogonal lift)**.** *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $\left(\bar{X}, \bar{Y}\right) \in \Gamma\left(T\bar{\mathscr{M}}\right) \times \Gamma\left(T\bar{\mathscr{M}}\right)$ be a couple of vector fields on the Platonic screen $\bar{\mathscr{M}}$ such that $\bar{X}$ and $\bar{Y}$ are not both spacelike. The couple of vector fields $(X, Y) \in \Gamma\,(T\mathscr{M}) \times \Gamma\,(T\mathscr{M})$ on $\mathscr{M}$ is said to be a generalised orthogonal lift of (X, Y) if X and Y satisfy the conditions:*
- *X and Y are ξ-invariant*
- *$\pi_* X = \bar{X}$ and $\pi_* Y = \bar{Y}$*
- *$g\,(X, Y) = -M^2\psi\,(X)\,\psi\,(Y)$.*

with M a fixed constant.

It can be checked that this substitution does not affect the validity of Lemmas 4.4.26 [6], and 4.4.27. Defining a generalised orthogonal operator by adapting Definition 4.4.28, in can be checked that Lemma 4.4.29 as well as Proposition 4.4.33 still hold, so that this new prescription can be used in order to induce a Koszul connection on $\bar{\mathscr{M}}$. The only sensible difference appears in Proposition 4.4.33, which gets modified into:

6. Notably, equalities (4.4.35) still hold.

Proposition 4.4.35. *Let $(\mathscr{M}, g, \xi)$ be a Platonic wave with Platonic screen $\bar{\mathscr{M}}$. Denote $(\mathscr{M}, \hat{g}, \xi)$ the Bargmann-Eisenhart wave conformally related to $(\mathscr{M}, g, \xi)$ and $\mathscr{S}\left(\mathscr{M}, \bar{\hat{\psi}}, \bar{\hat{h}}\right)$ the Augustinian structure induced by $(\mathscr{M}, \hat{g}, \xi)$. Furthemore, let $\bar{\hat{\nabla}}$ be the Newtonian connection induced by $\hat{g}$ on $\bar{\mathscr{M}}$ and denote $\left[\bar{\hat{N}}, \overset{\bar{\hat{N}}}{A}\right]$ the equivalence class characterising the Newtonian connection $\bar{\hat{\nabla}}$. Then the two following Propositions hold:*

- *The generalised orthogonal connection $\bar{\nabla}$ is a Platonic connection for the Aristotelian structure $\mathscr{A}\left(\bar{\mathscr{M}}, \bar{\psi}, \bar{h}\right)$ induced by $(\mathscr{M}, g, \xi)$.*
- *The Platonic connection $\bar{\nabla}$ is conformally related to the Newtonian connection compatible with $\mathscr{S}$ and characterised by the equivalence class $\left[\bar{\hat{N}}, \overset{\bar{\hat{N}}}{A} - \frac{M^2}{2}\bar{\Omega}\bar{\hat{\psi}}\right]$.*

Proof: The proof is mainly identical to the one of Proposition 4.4.33, except for the case where X is not ξ-orthogonal, which gets modified as follows:

- X is not ξ-orthogonal:

Since X is not ξ-orthogonal, it can be chosen to be satisfy $g(X, X) = -M^2\psi(X)\psi(X)$, so that (X, X) is a generalised orthogonal lift for the timelike vector field $\bar{X}$. Eq.(4.4.43) then becomes

$$\begin{aligned} \bar{\nabla}_{\bar{X}}\bar{X} &= \bar{\Omega}^{-2}\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}} + \frac{M^2}{2}\bar{h}\left(d\ln\bar{\Omega}\right)\bar{\psi}\left(\bar{X}\right)\bar{\psi}\left(\bar{X}\right) \\ &= \bar{\Omega}^{-2}\left(\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}} + \frac{M^2}{2}\bar{\hat{h}}\left(d\bar{\Omega}\right)\bar{\hat{\psi}}\left(\bar{\hat{X}}\right)\bar{\hat{\psi}}\left(\bar{\hat{X}}\right)\right). \end{aligned} \tag{4.4.47}$$

Since $\bar{\hat{\nabla}}$ is characterised by the equivalence class $\left[\bar{\hat{N}}, \overset{\bar{\hat{N}}}{A}\right]$ the term between brackets can be written in holonomic coordinates as

$$\bar{\hat{X}}\left[\bar{\hat{X}}^\lambda\right] + \left(\bar{\hat{\Gamma}}^\lambda_{\mu\nu} + \frac{M^2}{2}\bar{\hat{h}}^{\lambda\rho}\partial_\rho\bar{\Omega}\,\bar{\hat{\psi}}_\mu\bar{\hat{\psi}}_\nu\right)\bar{\hat{X}}^\mu\bar{\hat{X}}^\nu \tag{4.4.48}$$

where

$$\bar{\hat{\Gamma}}^\lambda_{\mu\nu} = \bar{\hat{N}}^\lambda\partial_{(\mu}\bar{\hat{\psi}}_{\nu)} + \frac{1}{2}\bar{\hat{h}}^{\lambda\rho}\left[\partial_\mu\overset{\bar{\hat{N}}}{\gamma}_{\rho\nu} + \partial_\nu\overset{\bar{\hat{N}}}{\gamma}_{\rho\mu} - \partial_\rho\overset{\bar{\hat{N}}}{\gamma}_{\mu\nu}\right] + \bar{\hat{h}}^{\lambda\rho}\bar{\hat{\psi}}_{(\mu}\overset{\bar{\hat{N}}}{F}_{\nu)\rho} \tag{4.4.49}$$

and $\overset{\bar{\hat{N}}}{F}_{\mu\nu} \equiv 2\partial_{[\mu}\overset{\bar{\hat{N}}}{A}_{\nu]}$. Now, performing the substitution $\overset{\bar{\hat{N}}}{A}_\mu \to \overset{\bar{\hat{N}}}{A}_\mu - \frac{M^2}{2}\bar{\Omega}\psi_\mu$, the coefficients (4.4.49) shift as

$$\bar{\hat{\Gamma}}^\lambda_{\mu\nu} \to \bar{\hat{\Gamma}}^\lambda_{\mu\nu} + \frac{M^2}{2}\bar{\hat{h}}^{\lambda\rho}\partial_\rho\bar{\Omega}\,\bar{\hat{\psi}}_\mu\bar{\hat{\psi}}_\nu.$$

The terms between brackets in expression (4.4.48) takes then the interpetation of the coefficients of a Newtonian connection $\bar{\tilde{\nabla}}$ preserving the Augustinian structure $\mathscr{S}\left(\mathscr{M}, \bar{\hat{\psi}}, \bar{\hat{h}}\right)$ and characterised by the equivalence class $\left[\bar{\hat{N}}, \overset{\bar{\hat{N}}}{A} - \frac{M^2}{2}\bar{\Omega}\bar{\hat{\psi}}\right]$.
Equation (4.4.47) takes then the form

$$\bar{\nabla}_{\bar{X}}\bar{X} = \bar{\Omega}^{-2}\bar{\tilde{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}}$$

for $\bar{X}$ timelike. A similar expression can be obtained when $\bar{X}$ is spacelike from eq.(4.4.44), since the supplementary term then vanishes.
This line of reasoning ensures that the following assertion holds

$$\forall \bar{X},\ \bar{\hat{X}} \in \Gamma\left(T\bar{\mathscr{M}}\right)\ /\ \bar{X} = \bar{\Omega}^{-1}\bar{\hat{X}}, \quad \bar{\nabla}_{\bar{X}}\bar{X} = \bar{\Omega}^{-2}\bar{\tilde{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}}.$$

According to Proposition 3.2.52, $\bar{\nabla}$ is then a Platonic connection conformally related to the Newtonian connection $\bar{\tilde{\nabla}}$ characterised by the equivalence class $\left[\bar{\hat{N}}, \overset{\bar{\hat{N}}}{A} - \frac{M^2}{2}\bar{\Omega}\bar{\hat{\psi}}\right]$.

□

Geometrising the Eisenhart-Lichnerowicz lift

The precedent Sections armed us with the necessary tools in order to take an informed look on the results of Section 1.4 and thus gain a more geometric understanding of the Eisenhart-Lichnerowicz lift. We start by emphasising that the Eisenhart-Lichnerowicz lift is bifid in nature, in the sense that it treats in a radically different way the ξ-orthogonal vector fields from the others. This was already appreciated at the level of the equations of motion for the ambient Lagrangian and was confirmed regarding parallelism projection in the last Sections. Before being more concrete, let us dissipate a slight ambiguity in the terminology between relativistic and nonrelativistic normalisation conditions. Let $(\mathscr{M}, g, \xi)$ be a Platonic wave and $X \in \Gamma(T\mathscr{M})$ be a projectable vector field. We denote by $\bar{X} \in \Gamma\left(T\bar{\mathscr{M}}\right)$ the projection of X on the Platonic screen. The relativistic affine geodesic normalisation condition reads $g(X, X) = -M^2$ where the constant M^2 is the squared mass of the relativistic particle. According to the sign of $g(X, X)$, the relativistic vector field X will be qualified by the epithet

$$\begin{cases} g(X,X) = -M^2 < 0 : \text{timelike} \\ g(X,X) = -M^2 = 0 : \text{null} \\ g(X,X) = -M^2 > 0 : \text{spacelike.} \end{cases}$$

Indeed, these relativistic denominations should not be confused with their nonrelativistic meanings as

$$\begin{cases} \bar{\psi}\left(\bar{X}\right) \neq 0 : \text{timelike} \\ \bar{\psi}\left(\bar{X}\right) = 0 : \text{spacelike.} \end{cases} \tag{4.4.50}$$

Now, since we will be dealing with conformally related structures, we introduce some last piece of terminology regarding nonrelativistic parameterisation. Let $\mathscr{A}\left(\bar{\mathscr{M}}, \bar{\psi}, \bar{\gamma}\right)$ be an Aristotelian structure conformally related to the Augustinian structure $\mathscr{S}\left(\bar{\mathscr{M}}, \bar{\bar{\psi}}, \bar{\bar{\gamma}}\right)$, so that $\bar{\psi} = \bar{\Omega}\bar{\bar{\psi}}$. A vector field $\bar{X}$ will be said

$$\begin{cases} \text{proper time parameterised if } \bar{\psi}\left(\bar{X}\right) = 1 \\ \text{absolute time parameterised if } \bar{\bar{\psi}}\left(\bar{X}\right) = 1. \end{cases}$$

We now consider a relativistic vector field $X \in \Gamma\left(T\mathscr{M}\right)$ which is affine geodesic parameterised, *i.e.* $g\left(X, X\right) = -M^2$ and investigate the two following cases:

- X is ξ-orthogonal ($\psi\left(X\right) = m = 0$):

  Since $X \in \operatorname{Ker}\psi$, the normalisation condition takes the form $\gamma\left(X, X\right) = -M^2$, where $\gamma \in \Gamma\left(\vee^2 \operatorname{Ker}\psi\right)$ is the relativistic spatial metric (*cf.* Definition 4.2.3). Since γ is positive semi-definite, the normalisation condition prevents X to be timelike. We thus distinguish between the two cases:

  - $M^2 = 0$:

    The normalisation condition thus reads $\gamma\left(X, X\right) = 0$, so that $X \sim \xi$. This case does not fall under the ambient scheme (since the projection of ξ on the Platonic screen vanishes) but rather describes the graviton worldlines (ξ has indeed been shown to be affine geodesic in Proposition 2.1.4).

  - $M^2 < 0$:

    This case has the nice feature to realise both meanings of the denomination "spacelike" (since $M^2 < 0$ and $\bar{\psi}\left(\bar{X}\right) = 0$). From an ambient viewpoint, since ψ is a conservor (*cf.* Definition 3.2.42), the quantity $\psi\left(X\right) = m$ is conserved along geodesics with respect to the Platonic Levi-Civita connection ∇. We conclude from this fact that any geodesic whose tangent vector lies in the wavefront worldvolume $\mathscr{W}_t$ at one point stays entirely in the hypersurface $\mathscr{W}_t$ (the wavefront worldvolumes can then be said to be totally geodesic submanifolds of $\mathscr{M}$ in this precise sense, although they are not Riemannian spaces). From a nonrelativistic point of view, we know from Section 4.4.2 that, since X is ξ-orthogonal , the geodesic equation $\nabla_X X = 0$ admits a well-defined projection on the absolute spaces Σ_t of the Platonic screen $\bar{\mathscr{M}}$ as $\overset{\bar{\gamma}}{\nabla}_{\bar{X}}\bar{X} = 0$, where $\overset{\bar{\gamma}}{\nabla}$ stands for the Levi-Civita connection associated to the spatial metric $\bar{\gamma} \in \Gamma\left(\vee^2 \operatorname{Ker} \bar{\psi}\right)$ obtained by projection of γ on the Platonic screen. We noted earlier (in Corollary 3.2.44), that the absolute spaces are totally geodesic submanifolds of $\bar{\mathscr{M}}$ in the same sense that the wavefront worldvolumes are with respect to $\mathscr{M}$.[7]

- X is not ξ-orthogonal ($\psi(X) = m \neq 0$):

  Since X is not ξ-orthogonal, we can assume without loss of generality that it is a relativistic field of observers $X \in FO(\mathscr{M})$, *i.e.* $\psi(X) = m = 1$. We now make the further distinction between the null and non-null cases:

  – $M^2 = 0$:

    Whenever the relativistic vector field X is null (so that X is a relativistic field of light-like observers), the affine geodesic normalisation condition takes the form $g(X, X) = 0$. In the light of the preceding Sections, one can reinterpret the normalisation condition as the orthogonal lift prescription of Definition 4.4.34. According to Proposition 4.4.33, the relativistic affine geodesic equation $\nabla_X X = 0$ admits a well defined projection on the Platonic screen as $\pi_*(\nabla_X X) = \bar{\nabla}_{\bar{X}}\bar{X} = 0$, where $\bar{\nabla}$ is the Platonic connection on $\bar{\mathscr{M}}$ induced by the Platonic Levi-Civita connection ∇ via the orthogonal prescription. Proposition 4.4.33 further ensures that the Platonic connection $\bar{\nabla}$ is conformally related (in the sense of Proposition 3.2.51) to the Newtonian connection $\bar{\hat{\nabla}}$ inherited from the Bargmann-Eisenhart wave $(\mathscr{M}, \hat{g}, \xi)$ conformally related to the Platonic wave $(\mathscr{M}, g, \xi)$ via the conformal factor Ω. This ensures that the following relation holds

    $$\bar{\nabla}_{\bar{X}}\bar{X} \quad = \quad \bar{\Omega}^{-2}\bar{\hat{\nabla}}_{\bar{\hat{X}}}\bar{\hat{X}} = 0$$

    with $\bar{\hat{X}} = \bar{\Omega}\,\bar{X}$ and where $\bar{\Omega} \in C^\infty\left(\bar{\mathscr{M}}\right)$ designates the projection of the conformal factor on $\bar{\mathscr{M}}$. Using the terminology introduced earlier, we can say that $\bar{X}$ is proper time parameterised while $\bar{\hat{X}}$ is absolute time parameterised. Summing up, the affine geodesic equation with respect to the Platonic Levi-Civita connection ∇ for relativistic field of light-like observers projects onto the affine geodesic equation with respect to the Newtonian connection $\bar{\hat{\nabla}}$ for absolute time parameterised non-relativistic field of observers. This constitutes a reinterpretation in terms of parallel transport of the dynamical equations (1.4.20)-(1.4.21) with $M^2 = 0$. Note that in this case, the nonrelativistic conformal factor $\bar{\Omega}$ does not appear in eq.(1.4.20)-

7. Note however that the situation is reversed in the sense that absolute spaces are Riemannian submanifolds embedded in the non-Riemannian manifold $\bar{\mathscr{M}}$

(1.4.21), as it should since null relativistic geodesics do not feel the relativistic conformal factor Ω.

– $M^2 \neq 0$:

In the non-null case, since X is chosen to be a relativistic field of observers, the relativistic affine geodesic normalisation condition can be rewritten in a bilinear form for X as

$$g\left(X, X\right) = -M^2 \psi\left(X\right) \psi\left(X\right) \tag{4.4.51}$$

where one recognises the generalised orthogonal lift prescription of Definition 4.4.34. Making use of Proposition 4.4.35, we can again rewrite the affine geodesic equation $\nabla_X X = 0$ as $\pi_*\left(\nabla_X X\right) = \bar{\nabla}'_{\bar{X}} \bar{X} = 0$, where $\bar{\nabla}'$ is the Platonic connection on $\bar{\mathscr{M}}$ induced by the Platonic Levi-Civita connection ∇ via the (generalised) orthogonal prescription. As showed in Proposition 4.4.35, the Platonic connection $\bar{\nabla}'$ is *not* conformally related to the Newtonian connection $\bar{\hat{\nabla}}$ induced by the Bargmann-Eisenhart wave $(\mathscr{M}, \hat{g}, \xi)$. Rather, $\bar{\nabla}'$ is conformally related to a *different* Newtonian connection $\bar{\hat{\nabla}}'$. Picking a nonrelativistic field of observers $\bar{N} \in FO\left(\bar{\mathscr{M}}\right)$, the gravitational potential 1-form $\overset{\bar{\hat{N}}}{A}{}'$ characterising $\bar{\hat{\nabla}}'$ differs from the one characterising $\bar{\hat{\nabla}}$, denoted $\overset{\bar{\hat{N}}}{A}$, by $\overset{\bar{\hat{N}}}{A}{}' = \overset{\bar{\hat{N}}}{A} - \frac{M^2}{2}\bar{\Omega}\bar{\hat{\psi}}$, where $\bar{\hat{\psi}} \in \Omega^1\left(\bar{\mathscr{M}}\right)$ is the closed absolute clock induced by the Bargmann-Eisenhart wave $(\mathscr{M}, \hat{g}, \xi)$ on $\bar{\mathscr{M}}$. Since $\overset{\bar{\hat{N}}}{A}{}'$ and $\overset{\bar{\hat{N}}}{A}$ differs by the 1-form $\bar{\hat{\psi}}$, they share the same Coriolis 1-form $\bar{A}_i \equiv \overset{N}{A}\left(e_i\right)$ but differ in their gravitational scalar potentials $\bar{U} \equiv -\overset{N}{A}\left(N\right)$. Now, similarly to the null case, the relation

$$\bar{\nabla}_{\bar{X}} \bar{X} \quad = \quad \bar{\Omega}^{-2} \bar{\hat{\nabla}}'_{\bar{\hat{X}}} \bar{\hat{X}}$$

holds where $\bar{\hat{X}}$ is absolute time parameterised. Writing the affine geodesic equation for $\bar{\hat{X}}$ with respect to $\bar{\hat{\nabla}}'$ in Brinkmann coordinates leads to eq.(1.4.20)-(1.4.21) where the gravitational scalar potential $\bar{U}$ [for the Newtonian connection inherited from $(\mathscr{M}, \hat{g}, \xi)$] is shifted to the effective gravitational scalar potential

$$\bar{V} = \bar{U} + \frac{M^2}{2}\bar{\Omega}.$$

Part II

Embedding nonrelativistic structures inside a Cartan geometry

Chapter 5

Cartan geometry

5.1 Klein geometry

As mentioned in the introduction, the nineteenth century has witnessed a conceptual revolution in the field of geometry with the independent discovery by Gauss, Bolyai and Lobachevsky of the Hyperbolic geometry in the plane. This first occurrence of a non-Euclidean geometry has then been followed, in a few decades, by the emergence of a number of new geometries such as Elliptic geometry, Affine geometry, Möbius geometry, Projective geometry, *etc.* In 1872, F. Klein published a pamphlet [143], in connection with his appointment to a chair in Erlangen (his manifesto is hence referred to as the *Erlangen Programm*, *cf.* [144] for historical details), whose input to the field of non-Euclidean geometries is twofold:

1. It proposed to subsum these different geometries to the study of Projective geometry as an unifying framework
2. It emphasised the role played by the underlying symmetry groups in the classification and relation between these different geometries.

We will focus in the present Chapter on the second of Klein's seminal propositions and on the subsequent generalisation by E. Cartan. Our presentation will rely heavily on the book [58] (*cf.* also [145, 146, 147, 148] for a mathematical viewpoint and [59, 60, 149, 150, 151] for a physical one).

One of the great insight of Klein was indeed the realisation that on each of the mentioned geometries, with space $\mathscr{M}$, there was a group G acting transitively (*cf.* Definition A.4.10), as illustrated by the two following examples:

Example 5.1.1 (Euclidean Plane).
Considering the Euclidean plane $\mathscr{M} = \mathbb{R}^2$, the group $G = ISO\,(2, \mathbb{R})$ defined as

$$ISO\,(2) = \left\{ \begin{pmatrix} 1 & 0 \\ v & \mathbf{R}\,(\theta) \end{pmatrix} \right\} \text{ with } \mathbf{R}\,(\theta) \equiv \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO\,(2, \mathbb{R}) \text{ and } v \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathscr{M}$$

acts transitively on $\mathscr{M}$ as

$$x \mapsto \begin{pmatrix} 1 & 0 \\ v & \mathbf{R}\,(\theta) \end{pmatrix} \cdot x = \mathbf{R}\,(\theta)\, x + v, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathscr{M}.$$

Furthermore, the stabiliser (or isotropy group, *cf.* Definition A.4.6) of the origin of $\mathbb{R}^2$ is given by

$$H = SO\,(2, \mathbb{R}) \equiv \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R}\,(\theta) \end{pmatrix}, \theta \in \mathbb{R} \right\}.$$

Example 5.1.2 (Hyperbolic Plane).
In this case, $\mathscr{M}$ can be chosen to be the Poincaré half-plane $\mathbb{H}^2 \equiv \{z = x + iy \in \mathbb{C} \,/\, y > 0\}$, on which the group $G = SL\,(2, \mathbb{R}) \equiv \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det A = 1 \right\}$ acts transitively according to

$$z \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The isotropy group of the point $i \in \mathbb{H}^2$ is given by $SO\,(2, \mathbb{R}) \equiv \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det A = 1, AA^T = 1 \right\}$.

This crucial observation of Klein allowed him to study a given geometry by shifting the emphasis from the space $\mathscr{M}$ to the group G. The following line of reasoning provides a precise way in which this can be achieved.

Let $\mathscr{M}$ be a manifold on which the Lie group G acts via the left-action $(G, \mathscr{M}) \to \mathscr{M}$: $(g, x) \mapsto g \cdot x$. The group G is assumed to act transitively, *i.e.* $\mathscr{M}$ is a homogeneous manifold (*cf.* Definition A.4.13). Let us particularise a point $x \in \mathscr{M}$ as "the origin" and denote $H_x \subset G$ the isotropy group (or stabiliser) of x. This choice of an origin defines a map $\pi_x : G \to \mathscr{M}$ as

$$\pi_x (g) = g \cdot x \quad \text{with } g \in G.$$

Proposition 5.1.3. *The map $\pi_x : G \to \mathscr{M}$ is onto.*

> **Proof:** Applying the transitivity condition of the action of G on $\mathscr{M}$ to the origin x leads to $\forall y \in \mathscr{M}$, $\exists\, g \in G / g \cdot x = y$, *i.e.* every point $y \in \mathscr{M}$ possesses an antecedent by the map π_x, so that π_x is onto. $\square$

Note that π_x is not generically one-to-one, since two elements g and g' of G differing by an element of the stabiliser of x (*i.e.* $g' = gh$, with $h \in H_x$) have the same image by π_x:

$$\pi_x \left(g'\right) = g' \cdot x = gh \cdot x = g \cdot x = \pi_x (g) \, .$$

Proposition 5.1.4. *Let $y \in \mathscr{M}$ and $g_0 \in G$ such that $\pi_x (g_0) = y$. Then the set of antecedents of y by π_x is the coset $g_0 H_x \equiv \{g_0 h / h \in H_x\}$, i.e.*

$$\pi_x^{-1} (y) = g_0 H_x.$$

> **Proof:** We first establish the sufficiency and then the necessity:
>
> $\underline{\Longrightarrow}$:
> Assuming $g' \in \pi_x^{-1} (y)$, one gets $g' \cdot x = y$. Furthermore, by definition of g_0, the following relation stands: $g_0 \cdot x = y$, so that $g' \cdot x = g_0 \cdot x$. Acting with g_0^{-1} on both sides leads to $g_0^{-1} g' \cdot x = x$ and $g_0^{-1} g'$ is hence seen to belong to H_x. Explicitly, there exists an element $h \in H_x$ such that $g_0^{-1} g' = h$, *i.e.* $g' = g_0 h$, which by Definition A.4.15, means that g' belongs to the coset $g_0 H_x$. We conclude that $g' \in \pi_x^{-1} (y)$ implies $g' \in g_0 H_x$.
>
> $\underline{\Longleftarrow}$:
> If $g' \in g_0 H_x$, then there exists an element $h \in H_x$ such that $g' = g_0 h$. Applying the map π_x to this identity leads to $\pi_x (g') = \pi_x (g_0 h) = g_0 h \cdot x = g_0 \cdot x = y$ so one is led to the conclusion that $g' \in g_0 H_x$ implies $g' \in \pi_x^{-1} (y)$. $\square$

From Proposition 5.1.4, one concludes that

Proposition 5.1.5. *The map $\rho_x : \mathscr{M} \to G / H_x$ defined by*

$$\rho_x (y) = \pi_x^{-1} (y) \, , \, y \in \mathscr{M}$$

is a G-space isomorphism.

Proof: We start by showing that ρ_x is bijective. Let $gH_x \in G/H_x$ be a coset, then gH_x admits necessarily the antecedent $y \equiv \pi_x(g)$, so that ρ_x is surjective. Now, let $y, y' \in \mathscr{M}$ and $g, g' \in G$ such that $\pi_x(g) = y$ and $\pi_x(g') = y'$. Assuming $\rho_x(y) = \rho_x(y')$ leads to $gH_x = g'H_x$ which is equivalent to $g' = gh$, for some $h \in H_x$, according to Proposition A.4.16. Hence, $y' = \pi_x(g') = \pi_x(gh) = \pi_x(g) = y$ and ρ_x is therefore injective.
Since ρ_x is a bijective map, all is left to do is to show the equivariance of ρ_x, *i.e.* that ρ_x satisfies $\rho_x(g \cdot y) = g\,\rho_x(y)$, $\forall y \in \mathscr{M}$ and $\forall g \in G$ (*cf.* Definition A.4.4). This is easily shown using the definition of ρ_x. Letting $g_0 \in G$ satisfy $\pi_x(g_0) = y$, one gets $\rho_x(g \cdot y) = \pi_x^{-1}(g \cdot y) = g\,g_0 H_x = g\,\rho_x(y)$, where the associativity of the G-action has been used. $\square$

The existence of the G-space isomorphism $\rho_x : \mathscr{M} \to G/H_x$ justifies Klein's viewpoint, according to which the study of a homogeneous manifold $\mathscr{M}$ boils down to the study of the set of (left) lateral classes G/H_x, where x is an arbitrary point of $\mathscr{M}$. Going back to our previous Examples, the Euclidean Plane can then be identified with the coset $\mathbb{R}^2 = ISO(2, \mathbb{R})/SO(2, \mathbb{R})$ while the Hyperbolic Plane reads $\mathbb{H}^2 = SL(2, \mathbb{R})/SO(2, \mathbb{R})$.[1] In the following, the term *Klein geometry* will either designate a pair (G, H) (where G is a Lie group and $H \subset G$ a closed subgroup) or the space $\mathscr{M} = G/H$.[2] Now, it can be shown

1. In the light of this result, the first of Klein's propositions, namely to subsum the study of Euclidean and non-Euclidean geometries (leaving aside Riemannian geometry) to the study of Projective geometry, can be understood using group-theoretical arguments. For instance, the structure group of the Euclidean Geometry is a subgroup of the affine group (which acts transitively on the space of Affine Geometry) being itself a subgroup of the group of projective transformations of $\mathbb{R}^d$ (the structure group of Projective Geometry). Therefore, the previous group hierarchy induces the following hierarchy of geometries:

$$\begin{array}{c}\text{Projective Geometry}\\ \bigcup\\ \text{Affine Geometry}\\ \bigcup\\ \text{Euclidean Geometry.}\end{array}$$

Going upward, one gains generality while going downward, one gets more structure. Hence, Euclidean space is more structured (*e.g.* angles and distance makes sense) than Projective Geometry in which such notions have no equivalent.

2. From now on, we drop the origin subscript for notational simplicity.

(*cf.* [58] §4.2) that the projection map $\pi : G \to G/H$ defines a H-principal bundle:

$$\begin{tikzcd} H \arrow[d, hook] \\ G \arrow[d, "\pi"] \\ G/H \end{tikzcd} \tag{5.1.1}$$

Notationwise, one designates by $L_g : G \to G : g' \mapsto gg'$ and $R_g : G \to G : g' \mapsto g'g$ the left and right actions of G on itself. The Lie algebras associated to the groups G and H will be denoted $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

Definition 5.1.6 (Maurer-Cartan form). *The Maurer-Cartan 1-form $\omega \in \Omega(G) \otimes \mathfrak{g}$ is a 1-form on G taking values in the Lie algebra $\mathfrak{g}$ and defined as the field of linear maps $\omega_g : T_gG \to T_eG \simeq \mathfrak{g}$ acting as*

$$\omega_g(v_g) = \left(L_{g^{-1}}\right)_* v_g, \ \textit{with } v_g \in T_gG. \tag{5.1.2}$$

It can be checked that the Maurer-Cartan 1-form satisfies the three following properties

1. $\omega_g : T_gG \to \mathfrak{g}$ is an isomorphism.
2. ω is right-equivariant by right-translation in G, *i.e.*

   $$\omega_{gg'}\left(R_{g'*}X_g\right) = \mathrm{Ad}\left(g'^{-1}\right)\omega_g\left(X_g\right), \ \forall\, g, g' \in G \text{ and } \forall\, X_g \in T_gG$$

   where $\mathrm{Ad} : G \to \mathfrak{g}$ is the adjoint representation of G on its Lie algebra $\mathfrak{g}$ (*cf.* Definition A.5.5).
3. $\omega_g\left(X_g^\sharp\right) = X$, $\forall X \in \mathfrak{g}$ with $X^\sharp \in \Gamma(TG)$ the fundamental vector field associated to the Lie algebra element X (*cf.* Definition A.5.10).

Anticipating the next Section, we note that these three properties will constitute the backbone of the generalisation of the Maurer-Cartan 1-form to Cartan's connections. In addition to these three properties, the Maurer-Cartan 1-form satisfies an additional crucial relation, known as the *structural equation*, which reads as

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \tag{5.1.3}$$

where $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ stands for the bracket of the Lie algebra $\mathfrak{g}$.

We will refer to the couple $(\mathfrak{g}, \mathfrak{h})$ as an *infinitesimal Klein pair*. Note that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, but is not generically an ideal, so that the quotient $\mathfrak{g}/\mathfrak{h}$ does not generically inherits a structure of Lie algebra. The projection of an element of $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{h}$ is denoted $\mathrm{pr} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Note that one can make use of the quotient projection in order to define an adjoint action $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to \mathrm{End}\,(\mathfrak{g}/\mathfrak{h})$ of H on the quotient vector space $\mathfrak{g}/\mathfrak{h}$ by making the following diagram commute:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}"] \arrow[d, "\mathrm{pr}"'] & \mathfrak{g} \arrow[d, "\mathrm{pr}"] \\ \mathfrak{g}/\mathfrak{h} \arrow[r, "\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}"] & \mathfrak{g}/\mathfrak{h} \end{tikzcd} \tag{5.1.4}$$

where $\mathrm{Ad} : H \to \mathrm{End}\,(\mathfrak{g})$ stands for the adjoint action of the subgroup H onto the Lie algebra $\mathfrak{g}$ (*cf.* Definition A.5.5).

Equipped with this representation, one is then able to construct generalised adjoint actions on the whole vector space of multilinear maps on $\mathfrak{g}/\mathfrak{h}$, denoted $\bigotimes \mathfrak{g}/\mathfrak{h} \otimes \bigotimes (\mathfrak{g}/\mathfrak{h})^*$. A typical example consists in defining an adjoint action on the dual vector space $(\mathfrak{g}/\mathfrak{h})^*$ as the transpose (*cf.* Definition A.1.6) of the adjoint action on $\mathfrak{g}/\mathfrak{h}$. This representation is referred in the literature as the contragredient adjoint representation and will be denoted $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}} : H \to \mathrm{End}\,(\mathfrak{g}/\mathfrak{h})^*$. The definition makes use of the original adjoint action as follows:

$$\left(\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,a\right)(x) = a\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)x\right) \tag{5.1.5}$$

with $x \in \mathfrak{g}/\mathfrak{h}$ and $a \in (\mathfrak{g}/\mathfrak{h})^*$. This construction can be easily generalised in order to define representations acting on higher-order vector spaces. Since this work deals with metric structures, we will be interested in defining an adjoint representation on the space of (contravariant and covariant) bilinear forms. The required construction is summarised in the following table with $x, y \in \mathfrak{g}/\mathfrak{h}$; $a, b \in (\mathfrak{g}/\mathfrak{h})^*$; $(\cdot,\cdot)^{-1} \in \vee^2 \mathfrak{g}/\mathfrak{h}$ and $(\cdot,\cdot) \in \vee^2\,(\mathfrak{g}/\mathfrak{h})^*$:

Symbol	Action	Definition
$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$	$H \to \mathrm{End}\,(\mathfrak{g}/\mathfrak{h})$	*cf.* Diagram 5.1.4
$\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}$	$H \to \mathrm{End}\,((\mathfrak{g}/\mathfrak{h})^*)$	$\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,a\,(x) = a\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)x\right)$
$\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}$	$H \to \mathrm{End}\left(\vee^2 \mathfrak{g}/\mathfrak{h}\right)$	$\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,(a,b)^{-1} = \left(\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)a, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)b\right)^{-1}$
$\bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}$	$H \to \mathrm{End}\left(\vee^2\,(\mathfrak{g}/\mathfrak{h})^*\right)$	$\bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,(x,y) = \left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)x, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)y\right)$

Table 5.1: Adjoint representations

Of special interest for the rest of this work will be elements of $\bigotimes \mathfrak{g}/\mathfrak{h} \otimes \bigotimes (\mathfrak{g}/\mathfrak{h})^*$ that are *adjoint invariant*, *i.e.* invariant under the generalised adjoint action (denoted for notational simplicity as $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to \bigotimes \mathfrak{g}/\mathfrak{h} \otimes \bigotimes (\mathfrak{g}/\mathfrak{h})^*)$. An element $\lambda \in \bigotimes \mathfrak{g}/\mathfrak{h} \otimes \bigotimes (\mathfrak{g}/\mathfrak{h})^*$ will then be said adjoint invariant if it satisfies $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,\lambda = \lambda$, $\forall h \in H$.

Example 5.1.7 (Killing form).

The Killing form [3] of a Lie group G is a symmetric bilinear form acting on the Lie algebra $\mathfrak{g}$ of G, *i.e.* $K \in \vee^2 \mathfrak{g}$, which is invariant under the adjoint action $\mathrm{Ad} : G \to \mathfrak{g}$. Whenever G is semisimple, the Killing form is nondegenerate and can then be used in order to define a distinguished element C of the universal envelopping algebra $U(\mathfrak{g})$, known as the quadratic Casimir element. The element C lies in the center of $U(\mathfrak{g})$, *i.e.* C commutes with all the generators of $\mathfrak{g}$. Alternatively, the quadratic Casimir can be interpreted as defining an Ad-invariant nondegenerate bilinear form on $\mathfrak{g}^*$ which obviously coincides with the inverse of the Killing form $K^{-1} \in \vee^2 \mathfrak{g}^*$.
Among the kinematical groups (*cf.* Appendix B.1) we will be interested in, the only semisimple groups are the (Anti)-de Sitter groups whose Lie algebra can be decomposed [4] as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $\mathfrak{h} \equiv \mathrm{Span}\, J_{\mu\nu}$ a Lie subalgebra and $\mathfrak{p} \equiv \mathrm{Span}\, P_\mu$ a vector space. The associated Killing form when expressed in this basis, reads $K(X, Y) = X^\mu Y^\nu \delta_{\mu\nu} + \frac{\Lambda}{2} X^{\mu\nu} Y^{\alpha\beta} \delta_{\mu\alpha}\delta_{\nu\beta}$ (the elements $X, Y \in \mathfrak{g}$ admits the decomposition $X \equiv X^\mu P_\mu + X^{\mu\nu} J_{\mu\nu}$ and $Y \equiv Y^\mu P_\mu + Y^{\mu\nu} J_{\mu\nu}$). The (Anti)-de Sitter quadratic Casimir element is expressed as $C = P_\mu P^{\mu *} + \frac{\Lambda}{2} J_{\mu\nu} J^{\mu\nu *}$ where Λ stands for the cosmological constant and $P^{\mu *}$, $J^{\mu\nu *}$ are the generators of the dual space $\mathfrak{g}^*$ associated respectively to P_μ, $J_{\mu\nu}$ via the (inverse) Killing form K^{-1}.
Since the (Anti)-de Sitter is semisimple and *symmetric* (*cf.* Section 5.3), the restriction of the inverse Killing form to the vector space $\mathfrak{p}$, denoted $(\cdot,\cdot)^{-1}_{\mathfrak{p}} \in \vee^2 \mathfrak{p}$ is nondegenerate and invariant under the projected adjoint action on contravariant bilinear forms on $\mathfrak{g}/\mathfrak{h} \sim \mathfrak{p}$ denoted $\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}} : H \to \mathrm{End}\left(\vee^2 \mathfrak{g}/\mathfrak{h}\right)$ (*cf.* Diagram 5.1.4 and Table 5.1).
Interestingly, even in cases where G is non-semisimple, quadratic elements belonging to the center of $U(\mathfrak{g})$ can be found so that one is still able to define (degenerate) Ad-invariant bilinear forms on $\mathfrak{g}^*$ and thus (possibly degenerate) $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant bilinear forms on $\mathfrak{g}/\mathfrak{h}^*$. This is best exemplified by the Poincaré algebra $\mathfrak{poin}$ for which the element $C \in U(\mathfrak{poin})$, defined as $C \equiv P_\mu P^{\mu *}$ (where $\{P_\mu\}$ are generators of $\mathfrak{poin}$ and $\{P^{\mu^*}\}$ the associated canonical dual basis), belongs to the center of $U(\mathfrak{poin})$. C in turn defines a nondegenerate contravariant bilinear form $(\cdot,\cdot)^{-1}_{\mathfrak{p}} \in \vee^2 \mathfrak{p}$, where $\mathfrak{p} \equiv \mathrm{Span}\, P_\mu$, acting as $(a, b) = a_\mu a_\nu \delta^{\mu\nu}$, where $a \equiv a_\mu P^{\mu *}$ and $b \equiv b_\mu P^{\mu *}$.

A related notion that will be proved useful for our purpose is the concept of *conformal adjoint invariance*. An element of $\lambda \in \bigotimes \mathfrak{g}/\mathfrak{h} \otimes \bigotimes (\mathfrak{g}/\mathfrak{h})^*$ will be be said confor-

3. The Killing form is sometimes referred to as the Cartan-Killing form in order to stress the importance of the contribution of Elie Cartan to the notion.

4. This is an H-module decomposition so that the (Anti)-de Sitter algebra is *reductive* (*cf.* Section 5.3).

mally invariant under the action $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$ if there exists a function $f : H \to \mathbb{R}$ such that $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,\lambda = f(h)\,\lambda$.

Example 5.1.8 (Conformal bilinear form).

Example 5.1.7 provided a nondegenerate contravariant bilinear form $(\cdot,\cdot)^{-1}_{\mathfrak{p}} \in \vee^2\,\mathfrak{p}$ acting on the vector space $\mathfrak{p} \equiv \mathrm{Span}\, P_\mu$ of the (Anti)-de Sitter and Poincaré algebras, which was argued to be $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant. Considering now the Weyl group G (whose Lie algebra also possesses $\mathfrak{p}$ as a subvector space) with homogeneous subgroup H, the bilinear form $(\cdot,\cdot)^{-1}_{\mathfrak{p}}$ transforms according to $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,(\cdot,\cdot)^{-1}_{\mathfrak{p}} = \lambda^2\,(\cdot,\cdot)^{-1}_{\mathfrak{p}}$ where $h \equiv \lambda\,\mathbf{R} \in H$, $\lambda \in \mathbb{R}$ and $\mathbf{R} \in O(d)$. Thus, $(\cdot,\cdot)^{-1}_{\mathfrak{p}}$ is conformally invariant under the action $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$ of the homogeneous Weyl group.

5.2 Cartan geometry

Klein's conceptual input left a scission in the field of geometry, with on one side what we will call Klein geometries, and on the other side Riemannian geometry. Indeed, while some of Klein geometries are Riemannian (*e.g.* Euclidean Plane), the vast majority is not (for instance, the notion of distance makes no sense in affine geometry). On the other hand, Riemannian geometries of non-constant curvature are not Kleinian. In the early 1920's, E.Cartan managed to reconcile these apparently incompatible theories by performing a common generalisation of Klein's and Riemann's theories. The content of this generalisation was neatly summarised by Cartan himself in the following quote (as cited in [58]):

> *In the wake of the movement of ideas which followed the general theory of relativity, I was led to introduce the notion of new geometries, more general than Riemannian geometry, and playing with respect to the different Klein geometries the same role as the Riemannian geometries play with respect to Euclidean space. The vast synthesis that I realised in this way depends of course on the ideas of Klein formulated in his celebrated Erlangen program while at the same time going far beyond it since it includes Riemannian geometry, which had formed a completely isolated branch of geometry, within the compass of a very general scheme in which the notion of group plays a fundamental role.*
>
> – E.Cartan, *in Selecta Jubilé Scientifique* (1939)

In the preface of [58] (*cf.* also [60]), Sharpe provides a useful pictorial description of

Cartan's approach, emphasising the twofold character of his generalisation:

$$\begin{tikzcd}
\text{Euclidean geometry} \arrow[r, "\text{Allow curvature}"] \arrow[d, "\text{Generalise symmetry group}"'] & \text{Riemannian geometry} \arrow[d, "\text{Generalise tangent space geometry}"] \\
\text{Klein geometry} \arrow[r, "\text{Allow curvature}"'] & \text{Cartan geometry}
\end{tikzcd}$$

Cartan geometries can thus be seen as curved Klein geometries, or equivalently as a non-Euclidean analog of Riemannian geometries. As we will comment extensively in the following, the beauty of the Cartanian approach lies in the deep relation these (curved, nonhomogeneous) geometries maintain with Lie groups, relation inherited from the homogeneous space modeling them.

The present Section is intended as an introduction to Cartan geometries. As an attempt to lighten the formal character of the exposition, we will pepper throughout the Section some applications to (pseudo)-Riemannian geometry, considered as a Cartan-Poincaré geometry. This series of examples will sketch a general procedure of which we will make use in the next Chapter in order to study nonrelativistic geometries.

Roughly speaking, a Cartan geometry can be defined as a principal H-bundle generalising the H-bundle of Diagram 5.1.1 endowed with a 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ generalising the Maurer-Cartan 1-form $\omega \in \Omega^1(G) \otimes \mathfrak{g}$. More precisely, the formal definition of a Cartan geometry (P, ω) with P a principal bundle and ω a Cartan connection can be stated as:

Definition 5.2.1 (Cartan geometry, *cf.* [58] Definition §5.3.1)**.** *A Cartan geometry $C = (P, \omega)$ modeled on the infinitesimal Klein pair $(\mathfrak{g}, \mathfrak{h})$ with structure group H (and $\mathfrak{h}$ the Lie algebra associated to H) consists of:*

1. *a smooth manifold $\mathscr{M}$ called the base*
2. *a principal right H-bundle P over $\mathscr{M}$ with projection $\pi : P \to \mathscr{M}$*
3. *a $\mathfrak{g}$-valued 1-form ω on P satisfying the following conditions:*
   - *(a) for each point $p \in P$, the linear map $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism*
   - *(b) $(R_h)^* \omega = \mathrm{Ad}\left(h^{-1}\right) \omega$, $\forall h \in H$*
   - *(c) $\omega\left(X^\sharp\right) = X$, $\forall X \in \mathfrak{h}$ where $X^\sharp \in VP$ designates the fundamental vector field associated to the Lie algebra element X.*

Several comments are in order. First, one observes that the Maurer-Cartan 1-form $\omega \in \Omega^1(G) \otimes \mathfrak{g}$ for the principal bundle defined by Diagram 5.1.1 satisfies the defining Axioms of a Cartan connection, so that (G, ω) qualifies as a Cartan geometry. Indeed, (G, ω) is the model geometry of which the Cartanian setup makes use in order to locally describe (P, ω).

Furthermore, one notes that the Axiom (a) imposes restrictions on the dimension of the base space (*i.e.* $\dim \mathscr{M} = \dim G/H$) which have no equivalent in the case of Ehresmann connections. This is needed in order to locally model the base space $\mathscr{M}$ by the Klein geometry G/H. An other important consequence of Axiom (a) is the fact that $\mathrm{Ker}\ \omega = 0$, so that generically, a Cartan geometry does not involve a notion of horizontality [5]. However, we will see that such a notion exists for a particular class of Cartan geometries (dubbed *reductive* geometries, *cf.* Section 5.3), allowing the definition of a Koszul connection on the base space.

Curvature & Torsion

The following Definitions and Propositions hold for a Cartan geometry $C = (P, \omega)$ modeled on $(\mathfrak{g}, \mathfrak{h})$ with structure group H.

Definition 5.2.2 (Curvature)**.** *The $\mathfrak{g}$-valued 2-form on P given by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ is called curvature.*

Comparison with the structural equation (5.1.3) reveals that the curvature Ω quantifies the local obstruction to P being isomorphic to the structure group G. Another nice feature of the Cartanian approach consists in further motivating the notion of torsion which is unified with the notion of curvature. This feature has the merit to add naturalness to the torsionfree condition.

Definition 5.2.3 (Torsion)**.** *The $\mathfrak{g}/\mathfrak{h}$-valued 2-form T defined by*

$$\begin{tikzcd} TP \times TP \arrow[r, "\Omega"] \arrow[rr, bend right, "T"'] & \mathfrak{g} \arrow[r, "pr"] & \mathfrak{g}/\mathfrak{h} \end{tikzcd}$$

is called torsion.

Proposition 5.2.4 (*cf.* [58] Corollary §5.3.10)**.** *The curvature 2-form $\Omega(X, Y)$ vanishes whenever X or Y is vertical.*

Proposition 5.2.5 (Bianchi identity, *cf.* [58] Lemma §5.3.30)**.** *The following identity holds:*

$$d\Omega = [\Omega, \omega]\,. \tag{5.2.6}$$

5. Recall, that, in an Ehresmann geometry, the horizontal distribution is precisely defined by $H_p = \mathrm{Ker}\ \omega_p$, with $p \in P$.

Proposition 5.2.6. *Let $f : P \to P$ be the map defined as $f(p) = R_{\psi(p)}p$ with $\psi : P \to H$ a map. The following relations hold:*

$$
\begin{aligned}
f^*\omega &= \mathrm{Ad}\left(\psi^{-1}\right)\omega + \psi^*\omega^H &\quad (5.2.7)\\
f^*\Omega &= \mathrm{Ad}\left(\psi^{-1}\right)\Omega &\quad (5.2.8)
\end{aligned}
$$

with $\omega^H \in \Omega^1(H) \otimes \mathfrak{h}$ the Maurer-Cartan 1-form on the subgroup H.

Proof: *cf.* the proof of Theorem §5.3.5 and Lemma §5.3.9 in [58]. □

Corollary 5.2.7. *The curvature 2-form transforms under the right-action as*

$$R_h^*\Omega = \mathrm{Ad}\left(h^{-1}\right)\Omega.$$

Proposition 5.2.8. *The vector space spanned by the values of the curvature 2-form is an H-submodule of $\mathfrak{g}$.*

Proof: Let V be the vector space spanned by values of Ω and let $v \in V$. Then, there exist $X_p, Y_p \in T_pP$ such that $v = \Omega_p(X_p, Y_p)$. Then we have $\mathrm{Ad}\left(h^{-1}\right)v = \mathrm{Ad}\left(h^{-1}\right)\Omega_p(X_p, Y_p) = R_h^*\Omega_{ph}(X_p, Y_p) = \Omega_{ph}(R_{h*}X_p, R_{h*}Y_p) \in V$, so that V is stable by the adjoint action. □

Isomorphisms

The following Theorem provides a concrete meaning to the idea of locally modeling the base space $\mathscr{M}$ by the model space G/H. The isomorphism thus defined will be ubiquitous in the rest of this work.

Theorem 5.2.9 (*cf.* [58] Theorem §5.3.15)**.** *For each point $p \in P$ with $\pi(p) = x$, there is a canonical linear isomorphism $\varphi_p : T_x\mathscr{M} \to \mathfrak{g}/\mathfrak{h}$ defined by*

$$
\begin{tikzcd}
T_pP \arrow[r, "\omega_p"] \arrow[d, "\pi_*"'] & \mathfrak{g} \arrow[d, "\mathrm{pr}"] \\
T_x\mathscr{M} \arrow[r, "\varphi_p", "\approx"'] & \mathfrak{g}/\mathfrak{h}.
\end{tikzcd}
\quad (5.2.9)
$$

Explicitly, the action of φ_p on $X_{\pi(p)} \in T_{\pi(p)}\mathscr{M}$ reads $\varphi_p\left(X_{\pi(p)}\right) = \mathrm{pr}\left(\omega_p\left(\tilde{X}_p\right)\right)$, where $\tilde{X}_p \in T_pP$ is a lift of $X_{\pi(p)}$, *i.e.* a vector field on P such that $\pi_*\tilde{X}_p = X_{\pi(p)}$. The isomorphism φ_p acting on $X_{\pi(p)} \in T_{\pi(p)}\mathscr{M}$ is canonical in the sense that it is independent of the choice of lift $\tilde{X}_p$. Indeed two lifts differ by a vertical part $\tilde{X}'_p - \tilde{X}_p \in V_p$ which is projected out by the action of $\mathrm{pr} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.

However, the isomorphism φ does depend on the point $p \in P$ and passing from p to $ph = R_h p$, φ transforms as $\varphi_{ph} = \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)\varphi_p$ while its inverse φ^{-1} transforms as $\varphi_{ph}^{-1} = \varphi_p^{-1}\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)$. These transformation properties follow readily from the equivariance of the Cartan connection ω.

In Section 5.1, the adjoint representation $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to \mathrm{End}\left(\mathfrak{g}/\mathfrak{h}\right)$ was used in order to generate various representations of the type $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to \bigotimes \mathfrak{g}/\mathfrak{h} \otimes \bigotimes \left(\mathfrak{g}/\mathfrak{h}\right)^*$. In the very same spirit, the isomorphism $\varphi_p : T_{\pi(p)}\mathscr{M} \to \mathfrak{g}/\mathfrak{h}$ can serve as a building block in order to construct isomorphisms between elements of $\bigotimes T_{\pi(p)}\mathscr{M} \otimes \bigotimes T^*_{\pi(p)}\mathscr{M}$ and $\bigotimes \mathfrak{g}/\mathfrak{h} \otimes \bigotimes \left(\mathfrak{g}/\mathfrak{h}\right)^*$. Despite vectors ($X_{\pi(p)} \in T_{\pi(p)}\mathscr{M}$), we will focus on linear forms ($\alpha_{\pi(p)} \in T^*_{\pi(p)}\mathscr{M}$) as well as on contravariant and covariant bilinear forms ($g^{-1}_{\pi(p)} \in \vee^2 T_{\pi(p)}\mathscr{M}$ and $g_{\pi(p)} \vee^2 T^*_{\pi(p)}\mathscr{M}$).

The next Proposition makes explicit the construction of a bijective correspondence between linear forms on $T_{\pi(p)}\mathscr{M}$ and $\mathfrak{g}/\mathfrak{h}$.

Proposition 5.2.10. *The transpose $\bar{\varphi}_p : T^*_{\pi(p)}\mathscr{M} \to \left(\mathfrak{g}/\mathfrak{h}\right)^*$ of the isomorphism $\varphi_p : T_{\pi(p)}\mathscr{M} \to \mathfrak{g}/\mathfrak{h}$ defined as*

$$\bar{\varphi}_p\left(\alpha_{\pi(p)}\right) x = \alpha_{\pi(p)}\left(\varphi_p^{-1}\left(x\right)\right)$$

where $\alpha_{\pi(p)} \in T^*_{\pi(p)}\mathscr{M}$ and $x \in \mathfrak{g}/\mathfrak{h}$, is an isomorphism. Passing from $p \in P$ to $ph = R_h p$, $\bar{\varphi}$ transforms as $\bar{\varphi}_{ph} = \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)\bar{\varphi}_p$ where $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}} : H \to End\left(\mathfrak{g}/\mathfrak{h}\right)^*$ is the adjoint contragredient representation. Its inverse $\bar{\varphi}^{-1}$ transforms as $\bar{\varphi}^{-1}_{ph} = \bar{\varphi}^{-1}_p\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)$.

Proof: The fact that $\bar{\varphi}$ is an isomorphism is obvious from its definition. Regarding its transformation law, one has:

$$\begin{aligned}
\bar{\varphi}_{ph}\left(\alpha_{\pi(p)}\right)(x) &= \alpha_{\pi(p)}\left(\varphi_{ph}^{-1}\left(x\right)\right)\\
&= \alpha_{\pi(p)}\left(\varphi_p^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)(x)\right)\right)\\
&= \bar{\varphi}_p\left(\alpha_{\pi(p)}\right)\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)(x)\right)\\
&= \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)\left(\bar{\varphi}_p\left(\alpha_{\pi(p)}\right)\right)(x)
\end{aligned}$$

with $\alpha_{\pi(p)} \in T^*_{\pi(p)}\mathscr{M}$ and $x \in \mathfrak{g}/\mathfrak{h}$, so that $\bar{\varphi}_{ph}\left(\alpha_{\pi(p)}\right) = \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)\left(\bar{\varphi}_p\left(\alpha_{\pi(p)}\right)\right)$. The transformation law for $\bar{\varphi}^{-1}$ follows readily. $\square$

The two next Propositions follow a very similar line of reasoning in order to construct isomorphisms between bilinear forms on $T_{\pi(p)}\mathscr{M}$ and $\mathfrak{g}/\mathfrak{h}$.

Proposition 5.2.11. *The map $\tilde{\varphi}_p : \vee^2 T_{\pi(p)}\mathscr{M} \to \vee^2 \mathfrak{g}/\mathfrak{h}$ defined as*

$$\tilde{\varphi}_p\left(g^{-1}_{\pi(p)}\right)(a,b) = g^{-1}_{\pi(p)}\left(\bar{\varphi}_p^{-1}(a), \bar{\varphi}_p^{-1}(b)\right) \tag{5.2.10}$$

where $g^{-1}_{\pi(p)} \in \vee^2 T_{\pi(p)}\mathscr{M}$ and $a, b \in (\mathfrak{g}/\mathfrak{h})^*$ is an isomorphism. Passing from $p \in P$ to $ph = R_h p$, $\tilde{\varphi}$ transforms as $\tilde{\varphi}_{ph} = \tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)\tilde{\varphi}_p$ where $\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}} : H \to End\left(\vee^2 \mathfrak{g}/\mathfrak{h}\right)$ is the adjoint bilinear contravariant representation. Its inverse $\tilde{\varphi}^{-1}$ transforms as $\tilde{\varphi}^{-1}_{ph} = \tilde{\varphi}^{-1}_p \tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)$.

Proposition 5.2.12. *The map $\bar{\tilde{\varphi}}_p : \vee^2 T^*_{\pi(p)}\mathscr{M} \to \vee^2 (\mathfrak{g}/\mathfrak{h})^*$ defined as*

$$\bar{\tilde{\varphi}}_p\left(g_{\pi(p)}\right)(x,y) = g_{\pi(p)}\left(\varphi_p^{-1}(x), \varphi_p^{-1}(y)\right) \tag{5.2.11}$$

where $g_{\pi(p)} \in \vee^2 T^*_{\pi(p)}\mathscr{M}$ and $x, y \in \mathfrak{g}/\mathfrak{h}$ is an isomorphism. Passing from $p \in P$ to $ph = R_h p$, $\bar{\tilde{\varphi}}$ transforms as $\bar{\tilde{\varphi}}_{ph} = \bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)\bar{\tilde{\varphi}}_p$ where $\bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}} : H \to End\left(\vee^2 (\mathfrak{g}/\mathfrak{h})^*\right)$ is the adjoint bilinear covariant representation. Its inverse $\bar{\tilde{\varphi}}^{-1}$ transforms as $\bar{\tilde{\varphi}}^{-1}_{ph} = \bar{\tilde{\varphi}}^{-1}_p \bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}(h)$.

The different isomorphisms precedently introduced are summarised in Table 5.2.

Tensor	Action	Definition
Vector $X_{\pi(p)} \in T_{\pi(p)}\mathscr{M}$	$\varphi_p : T_{\pi(p)}\mathscr{M} \to \mathfrak{g}/\mathfrak{h}$	*cf.* Theorem 5.2.9
1-form $\alpha_{\pi(p)} \in T^*_{\pi(p)}\mathscr{M}$	$\bar{\varphi}_p : T^*_{\pi(p)}\mathscr{M} \to (\mathfrak{g}/\mathfrak{h})^*$	$\bar{\varphi}_p\left(\alpha_{\pi(p)}\right)x = \alpha_{\pi(p)}\left(\varphi_p^{-1}(x)\right) \forall x \in \mathfrak{g}/\mathfrak{h}$
Contravariant bilinear form $g^{-1}_{\pi(p)} \in \vee^2 T_{\pi(p)}\mathscr{M}$	$\tilde{\varphi}_p : \vee^2 T_{\pi(p)}\mathscr{M} \to \vee^2 \mathfrak{g}/\mathfrak{h}$	$\tilde{\varphi}_p\left(g^{-1}_{\pi(p)}\right)(a,b) = g^{-1}_{\pi(p)}\left(\bar{\varphi}_p^{-1}(a), \bar{\varphi}_p^{-1}(b)\right)$ $\forall\, a, b \in (\mathfrak{g}/\mathfrak{h})^*$
Covariant bilinear form $g_{\pi(p)} \in \vee^2 T^*_{\pi(p)}\mathscr{M}$	$\bar{\tilde{\varphi}}_p : \vee^2 T^*_{\pi(p)}\mathscr{M} \to \vee^2 (\mathfrak{g}/\mathfrak{h})^*$	$\bar{\tilde{\varphi}}_p\left(g_{\pi(p)}\right)(x,y) = g_{\pi(p)}\left(\varphi_p^{-1}(x), \varphi_p^{-1}(y)\right)$ $\forall\, x, y \in \mathfrak{g}/\mathfrak{h}$

Table 5.2: Summary of the different isomorphisms φ

An important application of the preceding bijective maps φ consists in the construction of an isomorphism $\Psi : \Gamma(E) \to \mathscr{T}(V, \rho)$ defining a bijective correspondence between sections $f \in \Gamma(E)$ of the associated vector bundle $E = P \times_H (V, \rho)$ (*cf.* Definition A.6.11) and tensors of type $\mathscr{T}(V, \rho)$ on P (*cf.* Definition A.6.9).

Proposition 5.2.13 (*cf.* [58] Corollary §5.3.16)**.** *There is a bijective correspondence between vector fields on $\mathscr{M}$ and tensors of type $\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$ given by the isomorphism $\Psi : \Gamma(T\mathscr{M}) \to \mathscr{T}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right) : X \mapsto f_X$ as $\Psi(X)(p) \equiv f_X(p) = \varphi_p\left(X_{\pi(p)}\right)$.*

That the functions f_X transforms as $f_X(ph) = \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right) f_X(p)$ follows readily from the transformation law of φ_p.

In a similar fashion, one can define a bijection between 1-forms on $\mathscr{M}$ and tensors of type $\left((\mathfrak{g}/\mathfrak{h})^*, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$ as follows:

Proposition 5.2.14. *There is a bijective correspondence between 1-form fields on $\mathscr{M}$ and tensors of type $\left((\mathfrak{g}/\mathfrak{h})^*, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$ given by the isomorphism $\bar{\Psi} : \Omega^1(\mathscr{M}) \to \mathscr{T}\left((\mathfrak{g}/\mathfrak{h})^*, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right) : \alpha \mapsto \bar{f}_\alpha$ as $\bar{\Psi}(\alpha)(p) \equiv \bar{f}_\alpha(p) = \bar{\varphi}_p\left(\alpha_{\pi(p)}\right)$.*

Section	Tensor type	Isomorphism
Vector field $X \in \Gamma(T\mathscr{M})$	$\mathscr{T}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$	$\Psi(X)(p) = \varphi_p\left(X_{\pi(p)}\right)$
1-form field $\alpha \in \Omega^1(\mathscr{M})$	$\mathscr{T}\left((\mathfrak{g}/\mathfrak{h})^*, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$	$\bar{\Psi}(\alpha)(p) = \bar{\varphi}_p\left(\alpha_{\pi(p)}\right)$
Contravariant bilinear form $g^{-1} \in \Gamma\left(\vee^2 T\mathscr{M}\right)$	$\mathscr{T}\left(\vee^2 \mathfrak{g}/\mathfrak{h}, \tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$	$\tilde{\Psi}\left(g^{-1}\right)(p) = \tilde{\varphi}_p\left(g^{-1}_{\pi(p)}\right)$
Covariant bilinear form $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$	$\mathscr{T}\left(\vee^2 (\mathfrak{g}/\mathfrak{h})^*, \bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}\right)$	$\bar{\tilde{\Psi}}(g)(p) = \bar{\tilde{\varphi}}_p\left(g_{\pi(p)}\right)$

Table 5.3: Summary of the different isomorphisms Ψ

Example 5.2.15 (Pseudo-Riemannian geometry).

A (pseudo)-Riemannian geometry (or Lorentzian structure, *cf.* Definition 3.1.2) consists in a manifold $\mathscr{M}$ endowed with a pseudo-Riemannian metric $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$, *i.e.* a field $g : \mathscr{M} \to \vee^2 T^*\mathscr{M}$; $g : x \mapsto g_x$ of nondegenerate bilinear forms $g_x \in \vee^2 T^*_x\mathscr{M}$ of signature $(-, +, \ldots, +)$.
As mentioned in Example 5.1.7, the Poincaré group is endowed with an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant nondegenerate bilinear form $(\cdot, \cdot)_{\mathfrak{p}} : \vee^2 \mathfrak{p} \to \mathbb{R}$ acting on the vector space $\mathfrak{p} \equiv \mathrm{Span}\, P_\mu$ of the

Poincaré algebra. According to Proposition A.6.10, since $(\cdot,\cdot)_{\mathfrak{p}}$ is $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant, it induces a well-defined (constant) tensor on P of type $\left(\vee^2\,\mathfrak{p}^*, \bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\right)$. Acting on this constant tensor with $\bar{\tilde{\Psi}}^{-1} : \mathscr{T}\left(\vee^2\,\mathfrak{p}^*, \bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\right) \to \Gamma\left(\vee^2\, T^*\mathscr{M}\right)$ yields a nondegenerate covariant metric $g \in \Gamma\left(\vee^2\, T^*\mathscr{M}\right)$. Explicitly, at the point $x \in \mathscr{M}$ the bilinear form $g_x : T_x\mathscr{M} \vee T_x\mathscr{M} \to \mathbb{R}$ acts as $g_x\left(X_x, Y_x\right) \equiv \bar{\tilde{\varphi}}_p^{-1}\left(\cdot,\cdot\right)_{\mathfrak{p}}\left(X_x, Y_x\right) = \left(\varphi_p\left(X_x\right), \varphi_p\left(Y_x\right)\right)_{\mathfrak{p}}$ with $\pi\left(p\right) = x$ and X_x, $Y_x \in T_x\mathscr{M}$. The invariance of $(\cdot,\cdot)_{\mathfrak{p}}$ is required in order for g_x to be independent of the choice of representative $p \in \pi^{-1}\left(x\right)$ (recall that $\varphi_{ph}\left(X_x\right) = \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h^{-1}\right)\varphi_p\left(X_x\right)$). A Cartan-Poincaré geometry then induces a canonical nondegenerate metric on its base manifold so that it defines a pseudo-Riemannian geometry. A converse statement can be formulated (*cf.* Theorem §6.3.5. of [58]) as follows: there is a unique *torsion-free* Cartan-Poincaré geometry associated to a given pseudo-Riemannian geometry. This is an example of application for Cartan's method of equivalence (*cf.* [148]) which means to establish a correspondence between specified geometric features and particular Cartan geometries. The next example will provide yet another example of equivalence.

Example 5.2.15 again insists upon the fact that $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant structures on the vector space $\mathfrak{g}/\mathfrak{h} \sim \mathfrak{p}$ possess a favored status in the Cartanian approach. However, this does not mean that one must discard structures that are not $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant altogether. The following example explicits how functions of P taking values in $\mathfrak{p}$ which are not tensorial (*i.e.* do not satisfy the equivariant requirement) can be put in correspondence with fields on $\mathscr{M}$ provided a section $\sigma : \mathscr{M} \to P$ is given.

Example 5.2.16 (Conformal geometry).

A conformal geometry is here to be understood as a manifold endowed with a conformal class of metrics (*cf.* Definition 3.1.8). Such a conformal structure comes out naturally in Cartan-Weyl geometries, *i.e.* Cartan geometries modeled on the Klein pair $(\mathrm{Weyl}, \mathrm{Weyl}_0)$ where Weyl and Weyl_0 designate the Weyl and homogeneous Weyl group, respectively. For that purpose, one is naturally led to make use of the conformal bilinear form of Example 5.1.8 denoted $(\cdot,\cdot)_{\mathfrak{p}}$. Since $(\cdot,\cdot)_{\mathfrak{p}}$ is a constant element of $\vee^2\,\mathfrak{p}^*$ which is *not* adjoint invariant, Proposition A.6.10 provides $(\cdot,\cdot)_{\mathfrak{p}}$ to define a tensor on P, since it does not satisfy the equivariant requirement. Consequently, isomorphism $\bar{\tilde{\Psi}}^{-1} : \mathscr{T}\left(\vee^2\,\mathfrak{p}^*, \bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\right) \to \Gamma\left(\vee^2\, T^*\mathscr{M}\right)$ is inapplicable in this case so that $(\cdot,\cdot)_{\mathfrak{p}}$ does not induce a canonical metric structure on the base space $\mathscr{M}$. by defining $\overset{p}{g}_x\left(X_x, Y_x\right) \equiv \bar{\tilde{\varphi}}_p^{-1}\left(\cdot,\cdot\right)_{\mathfrak{p}}\left(X_x, Y_x\right) = \left(\varphi_p\left(X_x\right), \varphi_p\left(Y_x\right)\right)_{\mathfrak{p}}$ with $\pi\left(p\right) = x$ and X_x, $Y_x \in T_x\mathscr{M}$. The superscript p acts as a reminder of the fact that $\overset{p}{g}_x$ depends on the choice of the representative $p \in \pi^{-1}\left(x\right)$. Explicitly, under a change $p \to R_h p$, $\overset{p}{g}_x$ transforms as $\overset{ph}{g}_x = \bar{\tilde{\varphi}}_{ph}^{-1}\left(\cdot,\cdot\right)_{\mathfrak{p}} = \bar{\tilde{\varphi}}_p^{-1}\bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)\left(\cdot,\cdot\right)_{\mathfrak{p}} = \lambda^{-2}\bar{\tilde{\varphi}}_p^{-1}\left(\cdot,\cdot\right)_{\mathfrak{p}} = \lambda^{-2}\overset{p}{g}_x$, where $h \equiv \lambda\,\mathbf{R} \in \mathrm{Weyl}_0$, $\lambda \in \mathbb{R}$ and $\mathbf{R} \in O\left(d\right)$. The bilinear form induced at the point x by $(\cdot,\cdot)_{\mathfrak{p}}$ is then scaled by a conformal constant factor under a change of representative p.

In order to generalise the pointwise case, one needs to perform a choice of representative $p \in \pi^{-1}(x)$ at each point of the base manifold $x \in \mathscr{M}$, *i.e.* one needs a particular section $\sigma : \mathscr{M} \to P$ (*cf.* Definition A.6.3), so that $\overset{\sigma}{g}_x \equiv \bar{\bar{\varphi}}^{-1}_{\sigma(x)} (\cdot,\cdot)_{\mathfrak{p}}$. A change of section $\sigma \to \sigma'$ is carried out by making use of a map $h : \mathscr{M} \to \mathrm{Weyl}_0$, such that $\sigma'(x) = R_{h(x)}\sigma(x)$ (*cf.* Section 5.4). Decomposing the map h as $h(x) = \lambda(x)\,\mathbf{R}(x)$, where $\lambda : \mathscr{M} \to \mathbb{R}$ and $\mathbf{R} : \mathscr{M} \to O(d)$, the metric $\overset{\sigma}{g}$ defined as the field $x \mapsto \overset{\sigma}{g}_x$, $\forall x \in \mathscr{M}$ transforms under a change of section as $\overset{\sigma'}{g} = \lambda^{-2}(x)\,\overset{\sigma}{g}$. A Cartan-Weyl geometry endowed with a conformal bilinear form $(\cdot,\cdot)_{\mathfrak{p}}$ then defines an equivalence class of metrics $\mathcal{G} \equiv [g]$, in which two metrics g and g' differ by a conformal map λ. The gift of a section $\sigma : \mathscr{M} \to P$ singles out a representative of this conformal class while two conformally equivalent metrics are related by a change of section. Looking again, as in Example 5.2.15, for a converse statement, one realises that an additional structure is necessary in order to draw an equivalence with a torsion-free Cartan-Weyl geometry. Namely, one is led to introduce a *Weyl structure*, *i.e.* a conformal class of metrics $\mathcal{G}$ on $\mathscr{M}$ supplemented with a map $F : \mathcal{G} \to \Omega^1(\mathscr{M})$ satisfying $F\left(e^{\lambda}g\right) = F(g) - d\lambda$, $\forall\ \lambda : \mathscr{M} \to \mathbb{R}$ and $g \in \mathcal{G}$. A Weyl structure can then be seen as an equivalence class $[(g,\omega)]$ with $g \in \Gamma(T^*\mathscr{M}) \vee \Gamma(T^*\mathscr{M})$ and $\omega \in \Omega^1(\mathscr{M})$ in which two couples (g,ω) and (g',ω') are said equivalent if they satisfy the *Eichtransformation*:

$$
\begin{aligned}
g' &= e^{\lambda} g \\
\omega' &= \omega - d\lambda
\end{aligned}
$$

with $\lambda \in C^{\infty}(\mathscr{M})$. Such a 1-form ω can be defined as the part of the gauge-connection (*cf.* Section 5.4) taking values in dilatations. A Cartan-Weyl geometry then determines a canonical Weyl structure on $\mathscr{M}$ and conversely, there is a unique torsion-free Cartan-Weyl geometry giving rise to a particular Weyl structure (*cf.* Theorem §7.3.14 of [58]).

5.3 Reductive Cartan geometry

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. The Cartan geometry $C = (P,\omega)$ modeled on $(\mathfrak{g},\mathfrak{h})$ is said reductive if $\mathfrak{g}$ can be decomposed as a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $\mathfrak{p}$ an H-module[6]. The following commutation relations

$$
\begin{aligned}
[\mathfrak{h},\mathfrak{h}] &\subset \mathfrak{h} \\
[\mathfrak{p},\mathfrak{h}] &\subset \mathfrak{p}
\end{aligned}
$$

come from the fact that $\mathfrak{h}$ and $\mathfrak{p}$ are H-modules.

6. Obviously, $\mathfrak{h}$ is also an H-module.

As such, any form taking values in $\mathfrak{g}$ can be decomposed into $\mathfrak{h}$ and $\mathfrak{p}$ components. In particular, the Cartan connection can be written as $\omega = \omega^{\mathfrak{h}} + \omega^{\mathfrak{p}}$:

$$\begin{tikzcd}
 & & \mathfrak{h} \\
TP \arrow[r, "\omega"] \arrow[urr, "\omega^{\mathfrak{h}}"] \arrow[drr, "\omega^{\mathfrak{p}}"'] & \mathfrak{g} \arrow[ur, "\mathrm{pr}_{\mathfrak{h}}"'] \arrow[dr, "\mathrm{pr}_{\mathfrak{p}}"] & \\
 & & \mathfrak{p}.
\end{tikzcd}$$

The equivariance property of the Cartan connection as well as the fact that $\mathfrak{h}$ and $\mathfrak{p}$ are H-modules ensure that $\forall h \in H$:

$$\begin{aligned}
\omega^{\mathfrak{h}}\left(R_{h*}X\right) &= \mathrm{Ad}\left(h^{-1}\right)\omega^{\mathfrak{h}}\left(X\right) \\
\omega^{\mathfrak{p}}\left(R_{h*}X\right) &= \mathrm{Ad}\left(h^{-1}\right)\omega^{\mathfrak{p}}\left(X\right).
\end{aligned}$$

The curvature 2-form can also be decomposed along $\mathfrak{h}$ and $\mathfrak{p}$ parts as:

$$\Omega^{\mathfrak{h}}(X,Y) = d\omega^{\mathfrak{h}}(X,Y) + \left[\omega^{\mathfrak{h}}(X), \omega^{\mathfrak{h}}(Y)\right] + \mathrm{pr}^{\mathfrak{h}}\left(\left[\omega^{\mathfrak{p}}(X), \omega^{\mathfrak{p}}(Y)\right]\right) \tag{5.3.12}$$

$$\begin{aligned}
\Omega^{\mathfrak{p}}(X,Y) = {} & d\omega^{\mathfrak{p}}(X,Y) + \left[\omega^{\mathfrak{h}}(X), \omega^{\mathfrak{p}}(Y)\right] + \left[\omega^{\mathfrak{p}}(X), \omega^{\mathfrak{h}}(Y)\right] \\
& + \mathrm{pr}^{\mathfrak{p}}\left(\left[\omega^{\mathfrak{p}}(X), \omega^{\mathfrak{p}}(Y)\right]\right).
\end{aligned} \tag{5.3.13}$$

As $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{h}$, the $\mathfrak{p}$-part of the reductive curvature will be called torsion.

<u>N.B:</u> In the symmetric cases, which we will mostly be interested in, we have $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$ and then the curvature decomposes as:

$$\Omega^{\mathfrak{h}}(X,Y) = d\omega^{\mathfrak{h}}(X,Y) + \left[\omega^{\mathfrak{h}}(X), \omega^{\mathfrak{h}}(Y)\right] + \left[\omega^{\mathfrak{p}}(X), \omega^{\mathfrak{p}}(Y)\right] \tag{5.3.14}$$

$$\Omega^{\mathfrak{p}}(X,Y) = d\omega^{\mathfrak{p}}(X,Y) + \left[\omega^{\mathfrak{h}}(X), \omega^{\mathfrak{p}}(Y)\right] + \left[\omega^{\mathfrak{p}}(X), \omega^{\mathfrak{h}}(Y)\right]. \tag{5.3.15}$$

In such cases, the Bianchi identity (*cf.* Proposition 5.2.5) of a torsionfree Cartan geometry (*i.e.* $\Omega^{\mathfrak{p}} = 0$) can be decomposed into its $\mathfrak{p}$ and $\mathfrak{h}$ parts as:

$$\begin{cases}
\left[\Omega^{\mathfrak{h}}, \omega^{\mathfrak{p}}\right] = 0 & \text{:First Bianchi identity} \\
d\Omega^{\mathfrak{h}} = \left[\Omega^{\mathfrak{h}}, \omega^{\mathfrak{h}}\right] & \text{:Second Bianchi identity}
\end{cases} \tag{5.3.16}$$

As Sharpe puts it, reductive Cartan geometries posses a much richer structure that non reductive ones as they allow a notion of horizontality, which makes them closer to Ehresmann geometries. Before discussing the Horizontal distribution induced by a reductive

Cartan connection, we review some properties of the (canonical) vertical distribution.

Definition 5.3.1 (Vertical distribution)**.** *We define the vertical distribution as* $V = \{V_p\}$ *with* $V_p = \text{Ker }(\pi_{p*}) = \{X_p \in T_pP / \pi_{p*}(X_p) = 0\}$. *At each point* $p \in P$, *the vectors belonging to* V_p *are called vertical.*

Proposition 5.3.2. *The vertical distribution can be alternatively defined as* $V = \text{Ker }(\omega^{\mathfrak{p}})$.

Proof: In the reductive case, the following shortcut can be taken in the Diagram (5.2.9):

$$\begin{tikzcd} T_pP \arrow[d, "\pi_*"'] \arrow[dr, "\omega_p^{\mathfrak{p}}"] & \\ T_x\mathscr{M} \arrow[r, "\varphi_p", "\approx"'] & \mathfrak{p} \end{tikzcd} \tag{5.3.17}$$

and therefore $\omega_p^{\mathfrak{p}} = \varphi_p \circ \pi_*$. As φ_p is an isomorphism, we then have $\text{Ker }(\omega^{\mathfrak{p}}) = \text{Ker }(\pi_*)$. $\square$

Proposition 5.3.3. *The vertical distribution is involutive.*

Proof: Using expression (5.3.13), with $\omega^{\mathfrak{p}}(X) = \omega^{\mathfrak{p}}(Y) = 0$ as well as Proposition 5.2.4 ensures $d\omega^{\mathfrak{p}}(X,Y) = 0$ for all vertical vector fields X, Y and then $V = \text{Ker }(\omega^{\mathfrak{p}})$ is involutive according to Proposition A.3.5. $\square$

Definition 5.3.4 (Horizontal distribution)**.** *We define the horizontal distribution* $H : P \to TP : p \mapsto H_p$ *as* $H_p = \text{Ker }\left(\omega_p^{\mathfrak{h}}\right) = \left\{X_p \in T_pP / \omega_p^{\mathfrak{h}}(X_p) = 0\right\}$. *The vectors in it are therefore called horizontal vectors.*

Proposition 5.3.5. *The horizontal distribution is involutive iff* $\forall X, Y \in H$, *we have* $\Omega^{\mathfrak{h}}(X,Y) = \text{pr}^{\mathfrak{h}}([\omega^{\mathfrak{p}}(X), \omega^{\mathfrak{p}}(Y)])$.

Proof: The Proposition follows readily from expression (5.3.12) and Proposition A.3.5. $\square$

Proposition 5.3.6. *The following properties hold:*

1. *The horizontal distribution is equivariant, i.e.* $H_{ph} = R_{h*}H_p$, $\forall p \in P$ *and* $\forall h \in H$.
2. *The map* $\pi_* : H_p \to T_{\pi(p)}\mathscr{M}$ *is an isomorphism.*

> **Proof:** The proof of *1.* follows readily from the equivariance of $\omega^{\mathfrak{h}}$ while *2.* is immediate from the Diagram 5.3.17, recalling that $\omega_p^{\mathfrak{p}}$ furnishes an isomorphism between H_p and $\mathfrak{p}$. □

Definition 5.3.7 (Horizontal lift)**.** *Let $X \in \Gamma(T\mathscr{M})$ be a vector field on $\mathscr{M}$. The horizontal lift $\tilde{X} \in \Gamma(TP)$ of X is the unique vector field on P satisfying:*

1. $\pi_* \tilde{X} = X$
2. $\omega^{\mathfrak{h}}\left(\tilde{X}\right) = 0.$

The existence and uniqueness of $\tilde{X}$ come from the fact that $\pi_* : H_p \to T_{\pi(p)}\mathscr{M}$ is an isomorphism.

Definition 5.3.8 (Projectable vector field)**.** *A vector field $\tilde{X} \in \Gamma(TP)$ is said projectable if $\tilde{X}_{ph} - R_{h*}\tilde{X}_p \in V_{ph}$, $\forall p \in P$ and $\forall h \in H$.*

Proposition 5.3.9. *A projectable vector field admits a well-defined projection on the base space as $X \equiv \pi_* \tilde{X} \in \Gamma(T\mathscr{M})$.*

> **Proof:** Let us show that $\pi_*\tilde{X}$ is a well-defined vector field on $\mathscr{M}$. The map $\pi : P \to \mathscr{M}$ being surjective, $\pi_*\tilde{X}$ assigns a vector at each point $m \in \mathscr{M}$. However, being not injective, the assignment at a point m coming from two distinct points $p, p' \in \pi^{-1}(m)$ could differ. Explicitly, $\forall m \in \mathscr{M}$, one needs $\pi_*\tilde{X}_p = \pi_*\tilde{X}_{p'}$, $\forall p, p' \in \pi^{-1}(m)$. Since p and p' belongs to the same fiber, there exists $h \in H$ such that $p' = R_h p \equiv ph$. Using the projectability condition gives $\pi_*\tilde{X}_{p'} = \pi_*\tilde{X}_{ph} = \pi_*(\tilde{R}_{h*}\tilde{X}_p + V_{ph}) = \pi_*\tilde{X}_p$ and $X \equiv \pi_*\tilde{X}$ is then a well-defined vector field on $\mathscr{M}$. □

Definition 5.3.10 (Equivariant vector field)**.** *A vector field $\tilde{X} \in \Gamma(TP)$ is said equivariant if the following relation holds: $\tilde{X}_{ph} = R_{h*}\tilde{X}_p$, $\forall p \in P$ and $\forall h \in H$.*

Proposition 5.3.11. *A horizontal lift is equivariant.*

> **Proof:** Let $\tilde{X}_p$ be the horizontal lift of the vector $X_{\pi(p)}$. We have $\pi_*\left(R_{h*}\tilde{X}_p\right) = (\pi \circ R_h)_*\left(\tilde{X}_p\right) = \pi_*\left(\tilde{X}_p\right) = X_{\pi(p)}$ and $\omega_{ph}^{\mathfrak{h}}\left(R_{h*}\tilde{X}_p\right) = \mathrm{Ad}\left(h^{-1}\right)\omega_p^{\mathfrak{h}}\left(\tilde{X}_p\right) = 0$ so that $R_{h*}\tilde{X}_p$ is the horizontal lift of the vector $X_{\pi(p)}$ at ph and the uniqueness of the horizontal lift ensures $\tilde{X}_{ph} = R_{h*}\tilde{X}_p$. □

Proposition 5.3.12. *Let $X, Y \in \Gamma(T\mathscr{M})$ be two vector fields on $\mathscr{M}$ and denote $\tilde{X}, \tilde{Y} \in \Gamma(TP)$ their respective horizontal lifts. The horizontal part of $\left[\tilde{X}, \tilde{Y}\right]$ is the horizontal lift of $[X, Y]$.*

Proof: The vector field $\left[\tilde{X},\tilde{Y}\right]^H$ is by definition horizontal and furthermore satisfies $\pi_*\left(\left[\tilde{X},\tilde{Y}\right]^H\right)=\pi_*\left(\left[\tilde{X},\tilde{Y}\right]\right)=\left[\pi_*\tilde{X},\pi_*\tilde{Y}\right]=[X,Y]$, it is therefore the horizontal lift of $[X,Y]$. □

Cartan Koszul connection

As mentioned earlier, reductive Cartan geometries have the nice feature to allow the definition of a notion of parallelism on the base space.

Definition 5.3.13 (Cartan derivative)**.** *Let $X\in\Gamma\left(T\mathscr{M}\right)$ be a vector field of the base space $\mathscr{M}$ and denote $\tilde{X}\in\Gamma\left(TP\right)$ its horizontal lift. Let $f\in\Gamma\left(E\right)$ be a section of $E=P\times_H(V,\rho)$, the vector bundle associated to P whose sections are in one-to-one correspondence with tensors of type $\mathscr{T}\left(V,\rho\right)$ via the isomorphism Ψ (cf. Section A.6). The Cartan derivative of f along X is defined as $\nabla_X:\Gamma\left(E\right)\to\Gamma\left(E\right):\nabla_Xf=\Psi^{-1}\left(\tilde{X}\left[\Psi\left(f\right)\right]\right)$.*

Proposition 5.3.14. *The Cartan derivative is a Koszul connection.*

Proof: *cf.* Proposition §5.3.48 of [58]. □

Proposition 5.3.15. *Let $\lambda\in\bigotimes\mathfrak{p}\otimes\bigotimes\mathfrak{p}^*$ be an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant element of the vector space of multilinear maps on $\mathfrak{p}$. Then the section $f\in\Gamma\left(E\right)$ of the associated bundle $E=P\times_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}}\bigotimes\mathfrak{p}\otimes\bigotimes\mathfrak{p}^*$ defined as $f\equiv\Psi^{-1}\left(\lambda\right)$ is parallelised by the Cartan Koszul connection.*

Proof: The element λ being $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant, Proposition A.6.10 ensures that the function $P\to\bigotimes\mathfrak{p}\otimes\bigotimes\mathfrak{p}^*;p\mapsto\lambda$ is a tensor of type $\left(\bigotimes\mathfrak{p}\otimes\bigotimes\mathfrak{p}^*,\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$, so that the isomorphism $\Psi^{-1}:\mathscr{T}\left(\bigotimes\mathfrak{p}\otimes\bigotimes\mathfrak{p}^*,\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$ maps λ into a section $f\in\Gamma\left(E\right)$. Furthermore, if one lets $X\in\Gamma\left(T\mathscr{M}\right)$ be a vector field on $\mathscr{M}$, then, according to Definition 5.3.13, one gets the expression $\nabla_Xf=\Psi^{-1}\left(\tilde{X}\left[\Psi\left(f\right)\right]\right)$ where $\tilde{X}\in\Gamma\left(TP\right)$ stands for the horizontal lift of X. Replacing $\Psi\left(f\right)$ by its expression leads to $\nabla_Xf=\Psi^{-1}\left(\tilde{X}\left[\lambda\right]\right)=0$, so that f is parallel transported by the Cartan Koszul connection ∇. □

The following important Corollary follows in a straightforward way from the previous Proposition by considering $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant bilinear forms on $\mathfrak{p}$.

Corollary 5.3.16. *Any Cartan metric is compatible with the Cartan Koszul connection.*

Example 5.3.17 (Levi-Civita connection).

Example 5.2.15 pointed out the relation between Cartan geometries modeled on Poincaré and pseudo-Riemannian geometries. The preceding discussion allows to go further by endowing the base manifold of a Cartan-Poincaré geometry with a canonical Koszul connection compatible with the metric structure. In the case where the Cartan geometry is torsion-free, the Cartan derivative coincides with the Levi-Civita connection associated with the Cartan metric.

Proposition 5.3.18. *The Koszul torsion associated to the Cartan derivative is linked to the Cartan torsion via*

$$\Psi\left(T\left(X,Y\right)\right)=\Omega^{\mathfrak{p}}\left(\tilde{X},\tilde{Y}\right)-\mathrm{pr}^{\mathfrak{p}}\left(\left[\omega^{\mathfrak{p}}\left(X\right),\omega^{\mathfrak{p}}\left(Y\right)\right]\right)$$

with $X,Y\in\Gamma\left(T\mathscr{M}\right)$ *and where* $\tilde{X},\tilde{Y}\in\Gamma\left(TP\right)$ *are the horizontal lifts of* X *and* Y *respectively. The isomorphism* $\Psi:\Gamma\left(T\mathscr{M}\right)\to\mathscr{T}\left(\mathfrak{g}/\mathfrak{h},\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$ *is defined in Corollary 5.2.13.*

Proof: Starting from equation (A.9.13) and applying Ψ on both sides, one gets $\Psi\left(T\left(X,Y\right)\right)=\tilde{X}\left[f_Y\right]-\tilde{Y}\left[f_X\right]-f_{[X,Y]}$ where $f_X\left(p\right)=\varphi_p\left(X_x\right)$ (*cf.* Corollary 5.2.13). Using Diagram (5.3.17), one gets $f_X\left(p\right)=\omega^{\mathfrak{p}}_p\left(\tilde{X}_p\right)$, so that $\Psi\left(T\left(X,Y\right)\right)=\tilde{X}\left[\omega^{\mathfrak{p}}\left(\tilde{Y}\right)\right]-\tilde{Y}\left[\omega^{\mathfrak{p}}\left(\tilde{X}\right)\right]-\omega^{\mathfrak{p}}\left(\left[\tilde{X},\tilde{Y}\right]\right)$ and then $\Psi\left(T\left(X,Y\right)\right)=d\omega^{\mathfrak{p}}\left(\tilde{X},\tilde{Y}\right)$. Using expression (5.3.13) and the fact that $\omega^{\mathfrak{h}}\left(\tilde{X}\right)=\omega^{\mathfrak{h}}\left(\tilde{Y}\right)=0$ concludes the proof. □

Proposition 5.3.19. *The Koszul curvature associated to the Cartan derivative vanishes if and only if the horizontal distribution is involutive.*

Proof: One applies the isomorphism Ψ on both sides of the identity defining R and manipulates as in Proposition 5.3.18 :

$$\begin{aligned}
\Psi\left(R\left(X,Y;f\right)\right) &= \Psi\left(\nabla_X\nabla_Y f\right)-\Psi\left(\nabla_Y\nabla_X f\right)-\Psi\left(\nabla_{[X,Y]}f\right)\\
&= \tilde{X}\left[\Psi\left(\nabla_Y f\right)\right]-\tilde{Y}\left[\Psi\left(\nabla_X f\right)\right]-\widetilde{[X,Y]}\left[\Psi\left(f\right)\right]\\
&= \tilde{X}\left[\tilde{Y}\left[\Psi\left(f\right)\right]\right]-\tilde{Y}\left[\tilde{X}\left[\Psi\left(f\right)\right]\right]-\left[\tilde{X},\tilde{Y}\right]^H\left[\Psi\left(f\right)\right]\\
&= \left[\tilde{X},\tilde{Y}\right]\left[\Psi\left(f\right)\right]-\left[\tilde{X},\tilde{Y}\right]^H\left[\Psi\left(f\right)\right]\\
&= \left[\tilde{X},\tilde{Y}\right]^V\left[\Psi\left(f\right)\right]
\end{aligned}$$

where in the second step, Proposition 5.3.12 has been used. As one sees, R measures the failure of the involutivity of the horizontal distribution and so vanishes if and only if the horizontal distribution is involutive. For a relation to the $\mathfrak{h}$-valued part of the Cartan curvature, *cf.* Proposition 5.3.5. □

Automorphisms of a Cartan geometry

Definition 5.3.20 (ω-constant vector field)**.** *The vector field $X \in \Gamma(TP)$ defined on the principal bundle P is said to be ω-constant if there exists an element $x \in \mathfrak{g}$ such that $\omega_p(X_p) = x$, $\forall p \in P$.*

Proposition 5.3.21. *Let $X \in \Gamma(TP)$ be a ω-constant vector field such that $\omega(X) = x$ with $x \in \mathfrak{g}$, and $Y \in \Gamma(TP)$. The following relation holds:*

$$\mathcal{L}_X \omega(Y) = [\omega(Y), x] + \Omega(X, Y).$$

Whenever the vector field X is fundamental, we have $\mathcal{L}_X \omega(Y) = [\omega(Y), x]$.

Proof: Using Cartan's magic formula $\mathcal{L}_X \omega = d i_X \omega + i_X d\omega$ (where i denotes the interior product) and acting on a vector field $Y \in \Gamma(TP)$, we obtain:

$$\begin{aligned}
\mathcal{L}_X \omega(Y) &= d(\omega(X))(Y) + d\omega(X, Y) \\
&= d\omega(X, Y) \\
&= \Omega(X, Y) - [\omega(X), \omega(Y)] \\
&= [\omega(Y), x] + \Omega(X, Y)
\end{aligned}$$

where in the first and third steps the constancy of X has been used. Whenever X is fundamental, Proposition 5.2.4 imposes $\Omega(X, Y) = 0$ so that $\mathcal{L}_X \omega(Y) = [\omega(Y), x]$. $\square$

Proposition 5.3.22. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$ and $\mathcal{D} = \{\mathcal{D}_p\}$ a distribution defined on P as $\mathcal{D}_p = \{X_p \in T_pP / \omega_p(X_p) \in \mathfrak{g}'\}$. The distribution $\mathcal{D}$ is involutive if and only if $\Omega(X, Y) \in \mathfrak{g}'$ for all $X, Y \in \mathcal{D}$.*

Proof: According to Definition A.3.4, $\mathcal{D}$ is involutive if and only if $\omega(X) \in \mathfrak{g}'$, $\omega(Y) \in \mathfrak{g}' \Rightarrow \omega([X, Y]) \in \mathfrak{g}'$. Manipulating,

$$\begin{aligned}
\omega([X, Y]) &= X[\omega(Y)] - Y[\omega(X)] - d\omega(X, Y) \\
&= X[\omega(Y)] - Y[\omega(X)] + \frac{1}{2}[\omega(X), \omega(Y)] - \Omega(X, Y)
\end{aligned}$$

one sees that all the terms on the right-hand side but the last take value in $\mathfrak{g}'$. Therefore, $\omega([X, Y]) \in \mathfrak{g}'$ if and only if $\Omega(X, Y) \in \mathfrak{g}'$ for $X, Y \in \mathcal{D}$. $\square$

Definition 5.3.23 (Automorphism of a Cartan geometry)**.** *An automorphism of the Cartan geometry (P, ω) is a bundle automorphism of P which preserves ω, i.e. $\phi^* \omega = \omega$.*

Proposition 5.3.24. *Every equivariant vector field of P induces a bundle automorphism of P.*

Proof: Let $X \in \Gamma(TP)$ be a horizontal vector field of P with associated flow denoted by $\Phi_X : \mathbb{R} \times P \to P$. One introduces the path $\sigma_p : \mathbb{R} \to P$ defined by $\sigma_p(\lambda) = \Phi_\xi(\lambda, p)$. The vector field X is tangent to σ_p so that $X = \sigma_* D_t$. Introducing the path $\kappa : \mathbb{R} \to P$ defined as $\kappa(\lambda) = R_h(\sigma_p(\lambda))$ for some group element $h \in H$, one gets $\kappa_* D_t = R_{h*} \sigma_{p*} D_t = R_{h*} X = X$ where in the last step, Proposition 5.3.11 was used. The vector field X is then tangent to the path κ which can then be expressed as $\kappa(\lambda) = \Phi_X(\lambda, R_h(p))$. Denoting $\phi_\lambda : P \to P$ the map defined as $\phi_\lambda(p) = \Phi_X(\lambda, p)$, one obtains $R_h(\phi_\lambda(p)) = \phi_\lambda(R_h(p))$ so that the actions of ϕ_λ and R_h commute $\forall \lambda \in \mathbb{R}$ and $h \in H$. $\square$

5.4 Gauge version of Cartan geometry

We now introduce the base versions of the principal notions discussed in the preceding Section. This will allow us to make contact with the index notation, perhaps more common in the physics literature. The transition from bundle objects to their alter ego on the base is performed using sections. As we will see, a change of sections will be associated to gauge transformations of the base objects.

We will again consider a Cartan geometry $C = (P, \omega)$ modeled on $(\mathfrak{g}, \mathfrak{h})$ with base $\mathscr{M}$. One starts by defining a gauge connection as:

Definition 5.4.1 (Cartan gauge connection)**.** *Let $\overset{U}{\sigma} : U \to P$ be a section defined on the open subset $U \subset \mathscr{M}$. The 1-form $\overset{U}{\theta} \in \Omega^1(U) \otimes \mathfrak{g}$ defined by $\overset{U}{\theta} = \overset{U}{\sigma}{}^* \omega$ is said to be a Cartan gauge connection on U compatible with C. The pair $\left(U, \overset{U}{\theta}\right)$ is called a Cartan gauge.*

Proposition 5.4.2. *Let $\overset{U}{\sigma} : U \to P$ and $\overset{V}{\sigma} : V \to P$ be two sections with $U \cap V \neq \emptyset$. Let $h : \mathscr{M} \to H$ be the map defined by*

$$\begin{tikzcd} P \arrow[rr, "R_h"] & & P \\ & \mathscr{M} \arrow[ul, "\overset{U}{\sigma}"] \arrow[ur, "\overset{V}{\sigma}"'] & \end{tikzcd}$$

The Cartan gauge connections $\overset{U}{\theta}$ and $\overset{V}{\theta}$ associated to $\overset{U}{\sigma}$ and $\overset{V}{\sigma}$ respectively, satisfy on $U \cap V$, the following compatibility relation:

$$\overset{V}{\theta} = \mathrm{Ad}\left(h^{-1}\right) \overset{U}{\theta} + h^* \omega^H \tag{5.4.18}$$

with $\omega^H \in \Omega^1(H) \otimes \mathfrak{h}$ the Maurer-Cartan 1-form on the subgroup H.

Proof: Introducing the map $\psi : P \to H$ as

$$\begin{tikzcd} P \arrow[rr, "\psi"] & & H \\ & \mathscr{M} \arrow[ul, "\overset{U}{\sigma}"] \arrow[ur, "h"'] & \end{tikzcd}$$

and denoting f the map $f : P \to P : p \mapsto R_{\psi(p)} p$, one sees that the two sections $\overset{U}{\sigma}$ and $\overset{V}{\sigma}$ are related via $\overset{V}{\sigma} = f \circ \overset{U}{\sigma}$. Therefore, using relation (5.2.7) in Proposition 5.2.6, one gets $\overset{V}{\theta} = \overset{V}{\sigma}{}^* \omega = \overset{U}{\sigma}{}^* f^* \omega = \overset{U}{\sigma}{}^* \left(\mathrm{Ad}\left(\psi^{-1}\right) \overset{U}{\theta} + \psi^* \omega^H \right) = \mathrm{Ad}\left(h^{-1}\right) \overset{U}{\theta} + h^* \omega^H$. $\square$

Such a change of section is called a gauge transformation and will be denoted $\overset{U}{\theta} \Rightarrow_h \overset{V}{\theta}$. In the reductive case where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is an H-module decomposition, one obtains

$$\theta_V^{\mathfrak{p}} = \mathrm{Ad}\left(h^{-1}\right) \theta_U^{\mathfrak{p}} \tag{5.4.19}$$
$$\theta_V^{\mathfrak{h}} = \mathrm{Ad}\left(h^{-1}\right) \theta_U^{\mathfrak{h}} + h^* \omega^H. \tag{5.4.20}$$

Proposition 5.4.3. *The base version of the connection curvature is the curvature of the base version of the connection, i.e.*

$$\Theta_U \equiv \overset{U}{\sigma}{}^* \Omega = d\overset{U}{\theta} + \frac{1}{2} \left[\overset{U}{\theta}, \overset{U}{\theta} \right] \tag{5.4.21}$$

with Ω the curvature of ω and $\overset{U}{\theta} = \overset{U}{\sigma}{}^* \omega$.

Proof: The proof is straightforward from direct calculation. $\square$

Proposition 5.4.4. *The base curvature transforms under a gauge transformation $\overset{U}{\theta} \Rightarrow_h \overset{V}{\theta}$ as:*

$$\Theta_V = \mathrm{Ad}\left(h^{-1}\right) \Theta_U. \tag{5.4.22}$$

Proof: The setup is identical to the proof of Proposition 5.4.2 so that one can make use of the relation (5.2.8) : $\Theta_V = \overset{V}{\sigma}{}^*\Omega = \overset{U}{\sigma}{}^* f^*\Omega = \overset{U}{\sigma}{}^*\left(\mathrm{Ad}\left(\psi^{-1}\right)\Omega\right) = \mathrm{Ad}\left(h^{-1}\right)\Theta_U$. $\square$

Definition 5.4.5 (Gauge expression of a tensor)**.** *Let $\Phi : P \to V$ be a tensor of type (V,ρ) and (U,θ) a gauge corresponding to the section $\sigma : \mathscr{M} \to P$, then the map $\phi : U \to V$ defined as $\phi \equiv \Phi \circ \sigma$ is called the expression of the tensor Φ in the gauge (U,θ). We denote $\overset{U}{\mathscr{T}}(V,\rho)$ the set of expressions of tensors of type (V,ρ) living on the open set $U \in \mathscr{M}$.*

Proposition 5.4.6. *Let $\overset{U}{\phi} : U \to V$ be the expression of a tensor Φ of type (V,ρ) in the gauge $\left(U, \overset{U}{\theta}\right)$. Under a gauge transformation $\overset{U}{\theta} \Rightarrow_h \overset{V}{\theta}$ parameterised by $h : U \cap V \to H$, ϕ transforms as $\overset{V}{\phi} = \rho\left(h^{-1}\right)\overset{U}{\phi}$, where $\overset{V}{\phi}$ stands for the expression of Φ in the gauge $\left(V, \overset{V}{\theta}\right)$.*

Proof: Let $\overset{U}{\sigma} : U \to P$ and $\overset{V}{\sigma} : V \to P$ be the respective sections associated to the gauges $\left(U, \overset{U}{\theta}\right)$ and $\left(V, \overset{V}{\theta}\right)$. One has $\overset{V}{\phi} = \Phi \circ \overset{V}{\sigma} = \Phi \circ R_h \circ \overset{U}{\sigma} = \rho\left(h^{-1}\right)\Phi \circ \overset{U}{\sigma} = \rho\left(h^{-1}\right)\overset{U}{\phi}$. $\square$

For example, one can introduce the gauge expression of the canonical isomorphism $\Psi : \Gamma(E) \to \mathscr{T}(V,\rho)$ defining a bijective correspondence between sections $f \in \Gamma(E)$ of the associated vector bundle $E = P \times_H (V,\rho)$ and tensors of type $\mathscr{T}(V,\rho)$ on P as defined in Corollary 5.2.13 and Proposition 5.2.14.

Definition 5.4.7 (Base version of the isomorphism Ψ)**.** *Let $\left(U, \overset{U}{\theta}\right)$ be a gauge corresponding to the section $\overset{U}{\sigma} : \mathscr{M} \to P$. We denote by $\overset{U}{\Upsilon} : \Gamma(E) \to \overset{U}{\mathscr{T}}(V,\rho)$ the gauge expression of $\Psi : \Gamma(E) \to \mathscr{T}(V,\rho)$ defined by $\overset{U}{\Upsilon}(f) \equiv \Psi(f) \circ \overset{U}{\sigma}$ where $f \in \Gamma(E)$ is a section of E.*

Under a gauge transformation $\overset{U}{\theta} \Rightarrow_h \overset{V}{\theta}$, $\overset{U}{\Upsilon}(f)$ and its inverse transform according to

$$\begin{cases} \overset{V}{\Upsilon}(f) = \rho\left(h^{-1}\right)\overset{U}{\Upsilon}(f) \\ \overset{V}{\Upsilon}{}^{-1}(\phi) = \overset{U}{\Upsilon}{}^{-1}(\rho(h)\,\phi)\,. \end{cases}$$

Proposition 5.4.8. *Let (P,ω) be a reductive Cartan geometry modeled on the reductive infinitesimal Klein pair $(\mathfrak{g},\mathfrak{h})$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $\left(U, \overset{U}{\theta}\right)$ the Cartan gauge associated to the section $\sigma : \mathscr{M} \to P$. Then the isomorphism $\overset{U}{\Upsilon} : \Gamma(T\mathscr{M}) \to \overset{U}{\mathscr{T}}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$ when acting on a vector field $X \in \Gamma(T\mathscr{M})$ can be expressed as $\overset{U}{\Upsilon}(X) = \overset{U}{\theta}{}^{\mathfrak{p}}(X)$.*

Proof: Starting from Definition 5.4.7 for $\overset{U}{\Upsilon}$ and acting on a vector field $X \in \Gamma(T\mathscr{M})$ leads to $\overset{U}{\Upsilon}(X)(x) = \Psi(X)(\sigma(x)) = \varphi_{\sigma(x)}(X_x)$, where the expression of Ψ as defined in Corollary 5.2.13 has been used. Moreover, as represented on the Diagram 5.3.17, for all $p \in P$ such that $\pi(p) = x$, the isomorphism $\varphi_p : T_x\mathscr{M} \to \mathfrak{g}/\mathfrak{h}$ can be expressed in the reductive case as $\varphi_p(X_x) = \omega_p^{\mathfrak{p}}(X_p)$ where $X_p \in T_pP$ designates any (*i.e.* not necessarily horizontal) lift of $X_x \in T_x\mathscr{M}$. In particular, one can choose the lift $X_p = \sigma_* X_x$, so that the expression for $\overset{U}{\Upsilon}(X)(x)$ becomes $\overset{U}{\Upsilon}(X)(x) = \omega^{\mathfrak{p}}_{\sigma(x)}(\sigma_* X_x) = \overset{U}{\theta}{}^{\mathfrak{p}}_x(X_x)$. $\square$

Example 5.4.9 (Poincaré algebra).

Let (P, ω) be a reductive Cartan-Poincaré geometry modeled on the reductive infinitesimal Klein pair $(\mathfrak{poin}, \mathfrak{lor})$, where $\mathfrak{poin}$ and $\mathfrak{lor}$ designate the Poincaré and Lorentz algebras, respectively. The Poincaré algebra can be decomposed, as a vector space, into $\mathfrak{poin} = \mathfrak{lor} \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathrm{Span}\, P_a$ is the $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{SO}(d,1))$-invariant module corresponding to translations, where $\mathrm{SO}(d,1)$ stands for the Lorentz group. Let (U, θ) designate the Cartan gauge associated to the section $\sigma : \mathscr{M} \to P$. The part of the connection taking values into the H-module $\mathfrak{p}$ is denoted $\theta^{\mathfrak{p}} \in \Omega^1(\mathscr{M}) \otimes \mathfrak{p}$ and can be decomposed on a basis of $\mathfrak{p}$ as $\theta^{\mathfrak{p}} = \theta^a P_a$, where $\theta^a \in \Omega^1(\mathscr{M})$ is a set of $d+1$ 1-forms on $\mathscr{M}$. The set θ^a, with $a = 0 \ldots d$ is often called "coframe" in the physics literature. According to Proposition 5.4.8 the isomorphism $\Upsilon : \Gamma(T\mathscr{M}) \to \overset{U}{\mathscr{T}}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$, introduced in Definition 5.4.7, acts on a vector field $X \in \Gamma(T\mathscr{M})$ as $\Upsilon(X) = \theta^a(X) P_a$. Now the inverse isomorphism $\Upsilon^{-1} : \overset{U}{\mathscr{T}}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right) \to \Gamma(T\mathscr{M})$ can be expressed as $\Upsilon^{-1}(\phi) = e_a P^{a*}(\phi)$, with $\phi \in \overset{U}{\mathscr{T}}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$ a tensor on $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{p}$ and $e_a \in \Gamma(T\mathscr{M})$ a set of $d+1$ vector fields, sometimes referred to as the "frame".

The inverse isomorphism $\Upsilon^{-1} : \overset{U}{\mathscr{T}}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right) \to \Gamma(T\mathscr{M})$ can now be used to define the isomorphism $\bar{\Upsilon} : \Omega^1(\mathscr{M}) \to \overset{U}{\mathscr{T}}\left((\mathfrak{g}/\mathfrak{h})^*, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$ mapping bijectively 1-form fields on $\mathscr{M}$ to tensors taking values into the dual vector space $\mathfrak{p}^* \simeq (\mathfrak{g}/\mathfrak{h})^*$ and transforming according to the contragredient representation $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}} : H \to \mathrm{End}((\mathfrak{g}/\mathfrak{h})^*)$ defined in 5.1. Given a 1-form field $\alpha \in \Omega^1(\mathscr{M})$ and a tensor $\phi \in \overset{U}{\mathscr{T}}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$ on $\mathfrak{g}/\mathfrak{h}$, the isomorphism $\bar{\Upsilon}$ is defined as $\bar{\Upsilon}(\alpha)\phi = \alpha\left(\Upsilon^{-1}(\phi)\right)$ and can then be expressed using the precedently introduced basis decomposition as $\bar{\Upsilon}(\alpha) = \alpha(e_a) P^{a*}$. Accordingly, the inverse isomorphism $\bar{\Upsilon}^{-1} : \overset{U}{\mathscr{T}}\left((\mathfrak{g}/\mathfrak{h})^*, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right) \to \Omega^1(\mathscr{M})$ acts on $\bar{\phi} \in \overset{U}{\mathscr{T}}\left((\mathfrak{g}/\mathfrak{h})^*, \bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$ as $\bar{\Upsilon}^{-1}\left(\bar{\phi}\right)(X) = \bar{\phi}(\Upsilon(X))$, with $X \in \Gamma(T\mathscr{M})$, so that $\bar{\Upsilon}^{-1}$ can be expressed as $\bar{\Upsilon}^{-1}\left(\bar{\phi}\right) = \theta^a(X)\,\bar{\phi}(P_a)$.

According to their definitions, the frame $e_a \equiv \Upsilon^{-1}(P_a)$ and coframe $\theta^a \equiv \bar{\Upsilon}^{-1}(P^{a*})$ have to satisfy the following relations: $\theta^a(e_b) = \theta^a\left(\Upsilon^{-1}(P_b)\right) = \bar{\Upsilon}(\theta^a)(P_b) = P^{a*}(P_b) = \delta^a_b$. Similarly, if we let $X \in \Gamma(T\mathscr{M})$ be a vector field on $\mathscr{M}$, the consistency relation $\Upsilon^{-1}(\Upsilon(X)) = X$ imposes the following equality: $\Upsilon^{-1}(\Upsilon(X)) = \Upsilon^{-1}(\theta^a(X)P_a) = \theta^a(X)e_a = X$. Expressing the frame and coframe in an holonomic basis as $e_a \equiv e_a^\mu \partial_\mu$ and $\theta^a \equiv \theta^a_\mu dx^\mu$ these two consistency relations read

$$\begin{cases} \theta^a_\mu e^\mu_b = \delta^a_b \\ \theta^a_\nu e^\mu_a = \delta^\mu_\nu \end{cases} \tag{5.4.23}$$

so that frame and coframe are seen to be inverse.

Proposition 5.4.10. *Let (U, θ) be a gauge for a reductive Cartan geometry (P, ω) modeled on the infinitesimal Klein pair $(\mathfrak{g}, \mathfrak{h})$, $X \in \Gamma(TU)$ a vector field on U and $f \in \Gamma(E)$ a section of the associated vector bundle $E = P \times_H (V, \rho)$. The expression in the gauge (U, θ) of the Koszul connection $\nabla_X : \Gamma(E) \to \Gamma(E)$ reads:*

$$\nabla_X f = \Upsilon^{-1}\left\{X[\Upsilon(f)] - \rho_*\left(\theta^{\mathfrak{h}}(X)\right)\Upsilon(f)\right\} \tag{5.4.24}$$

where $\Upsilon : \Gamma(E) \to \overset{U}{\mathscr{T}}(V, \rho)$ is the isomorphism defined in 5.4.7, $\theta^{\mathfrak{h}}$ designates the part of the gauge connection taking values in the homogeneous Lie subalgebra $\mathfrak{h}$ and $\rho_* : \mathfrak{h} \to$ $\mathrm{End}(V)$ is the derivative at the identity of the representation $\rho : H \to \mathrm{Gl}(V)$.

Proof: *cf.* [58] Proposition §5.3.49. □

Example 5.4.11 (Poincaré Koszul connection).

Let us carry on with our study of the Cartan-Poincaré geometry initiated in Example 5.4.9 by applying the preceding expression to a vector field $Y \equiv Y^\mu \partial_\mu \in \Gamma(TU)$. The isomorphism Υ maps Y to a $\overset{U}{\mathscr{T}}\left(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\right)$ tensor with $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ the adjoint representation on $\mathfrak{g}/\mathfrak{h}$ while ρ_* stands [7] for the pushforward of $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$ denoted $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$. Furthermore, the $\mathfrak{h}$-part of the gauge connection is expressed on the canonical basis of $\mathfrak{h} = \mathrm{Span}\, J_{ab}$ as $\theta^{\mathfrak{h}} \equiv \omega^{ab} J_{ab}$ where $\omega^{ab} \in \Omega^1(U)$ is a set of $\frac{d(d-1)}{2}$ 1-form fields on U antisymmmetric in (a, b). The set ω^{ab} is often reffered to as the "spin-connection" in the physics literature.

Applying formula 5.4.24 leads then to the familiar expression:

$$\nabla_X Y = X^\mu \left(\partial_\mu Y^a + \omega_\mu^{\ a}{}_b Y^b\right) e_a,$$

with $Y^a \equiv Y^\mu \theta^a_\mu$. Making contact with the expression of the Koszul connection in an holonomic basis $\nabla_X Y \equiv X^\mu \left(\partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu} Y^\nu\right) \partial_\lambda$ leads to the following relation between

7. We choose the normalisation: $\rho_* = -\frac{i}{2}\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}$.

coefficients $\Gamma^{\lambda}_{\mu\nu}$ and the spin-connection:

$$\partial_\mu \theta^a_\nu + \omega^{\ a}_{\mu\ b}\, \theta^b_\nu - \Gamma^{\lambda}_{\mu\nu}\, \theta^a_\lambda = 0. \tag{5.4.25}$$

This expression is sometimes referred to as the "first vielbein postulate" (*cf.* *e.g.* [10] §1.4) although this term seems somewhat improper since it is derived as a direct consequence of the definition for the coefficients $\Gamma^{\lambda}_{\mu\nu}$. One can make use of this expression in order to express the the coefficients $\Gamma^{\lambda}_{\mu\nu}$ in terms of the (co)-frame and spin connection in the gauge (U, θ):

$$\Gamma^{\lambda}_{\mu\nu} = e^{\lambda}_a\, \partial_\mu \theta^a_\nu + \omega^{\ a}_{\mu\ b}\, e^{\lambda}_a\, \theta^b_\nu. \tag{5.4.26}$$

By definition of $\Gamma^{\lambda}_{\mu\nu}$, the obtained expression is gauge invariant. In the case of a torsion-free geometry, the Christoffel symbols are symmetric in (μ, ν), so that $e^{\lambda}_a\, \partial_{[\mu}\theta^a_{\nu]} + \omega^{\ a}_{[\mu\ b}\, e^{\lambda}_a\, \theta^b_{\nu]} = 0$. Solving this constraint allows to express the spin connection $\omega^a{}_b$ solely in terms of the frame and coframe as :

$$\omega^{ab}_\mu = e^{\nu[a}\, \partial_{[\mu}\theta^{b]}_{\nu]} - e^{\rho[a}\, e^{\sigma b]}\, \partial_\rho \theta^c_\sigma\, \theta_{c\mu}.$$

Plugging back into equation (5.4.26) and manipulating a bit gives the, manifeslty gauge-invariant, expression (3.1.1) of the Christoffel symbols for the Levi-Civita connection:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right].$$

Example 5.4.12 (Weyl Koszul connection).

We here follow the same steps as in Example 5.4.11 in the case of a Cartan-Weyl geometry. Choosing the canonical basis for the homogeneous Weyl subalgebra as $\mathfrak{h} = \text{Span}\ J_{ab}, D$, one decomposes the homogeneous part of the gauge connection as $\theta^{\mathfrak{h}} = \omega^{ab} J_{ab} + \omega D$. Besides the spin-connection ω^{ab}, the homogeneous Weyl gauge-connection contains an additional 1-form $\omega \in \Omega^1(U)$ corresponding to dilatations. The Weyl Koszul connection then takes the form $\nabla_X Y = X^\mu \left(\partial_\mu Y^a + \omega^{\ a}_{\mu\ b} Y^b + \frac{1}{2}\omega_\mu Y^a\right) e_a$, where $X, Y \in \Gamma(T\mathscr{M})$ and $Y^a \equiv Y^\mu \theta^a_\mu$. The "first vielbein postulate" differs then from the Poincaré case as:

$$\Gamma^{\lambda}_{\mu\nu} = e^{\lambda}_a \left[\partial_\mu \theta^a_\nu + \omega^{\ a}_{\mu\ b}\, \theta^b_\nu + \frac{1}{2}\omega_\mu \theta^a_\nu\right]. \tag{5.4.27}$$

The torsion-free requirement again allows to solve the spin-connection in terms of the (co)-frame, with the difference that ω^{ab} also picks up a dependence in ω, which is not constrained by the torsion-free condition. The explicit expression for the torsion-free spin connection reads:

$$\omega_\mu^{ab} = e^{\nu[a}\, \partial_{[\mu}\theta_{\nu]}^{b]} - e^{\rho[a}\, e^{\sigma b]}\, \partial_\rho \theta_\sigma^c\, \theta_{c\mu} + \frac{1}{2}\left[e^{\nu[a}\, \omega_{[\mu}\theta_{\nu]}^{b]} - e^{\rho[a}\, e^{\sigma b]}\, \omega_\rho \theta_\sigma^c\, \theta_{c\mu}\right]$$

which, once plugged back into (5.4.27) leads to the expression (3.1.4) for Christoffel symbols

$$\Gamma_{\mu\nu}^\lambda \quad = \quad \frac{1}{2} g^{\lambda\rho}\left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right] + \frac{1}{2}\left[\delta_\mu^\lambda \omega_\nu + \delta_\nu^\lambda \omega_\mu - g^{\lambda\rho} g_{\mu\nu}\, \omega_\rho\right].$$

It has been noted (*cf.* 3.1) that this expression is invariant under an *Eichtransformation*

$$g' = e^\lambda g \qquad \text{and} \qquad \omega' = \omega - d\lambda \tag{5.4.28}$$

parameterised by the function $\lambda \in C^\infty(\mathscr{M})$. This invariance can be understood in the present formalism as the gauge-invariance of the Cartan Koszul connection (*i.e.* invariance under a change of sections $\overset{U}{\sigma} \Rightarrow_h \overset{V}{\sigma}$). An Eichtransformation is just a particular case of a gauge-transformation in a Cartan-Weyl geometry when the gauge-defining map $h : \mathscr{M} \to H$ takes values only in the dilatation part of the Weyl homogeneous group.

The integrable case (*cf.* Definition 3.1.10) can be recovered by supplementing the torsion-free condition with the requirement that the part of the gauge-curvature taking values in the dilatation vanishes, *i.e.* $\Omega^D = d\omega = 0$.

Proposition 5.4.13 (Relation between curvature tensor and gauge curvature)**.**

$$\overset{\sigma}{\Upsilon}\left(R\left(X, Y; f\right)\right) = \rho_*\left(\Theta^{\mathfrak{h}}\left(X, Y\right)\right) \overset{\sigma}{\Upsilon}\left(f\right) \tag{5.4.29}$$

Example 5.4.14 (Riemann tensor and Bianchi identities). On the one hand, one can decompose the gauge curvature of a torsionfree Cartan-Poincaré geometry on the canonical basis $\mathfrak{h} = \text{Span } J_{ab}$ as:

$$\Theta^{\mathfrak{h}}\left(X, Y\right) \equiv R^{ab}\left(X, Y\right) J_{ab} = R^{ab}{}_{\mu\nu} X^\mu Y^\nu J_{ab}.$$

On the other hand, the curvature tensor for the Cartan Koszul connection (*i.e.* the Riemann tensor) can be expressed in an holonomic basis as:

$$R^\lambda_{\rho\mu\nu} = R^{ab}{}_{\mu\nu} \theta_{b\rho} e_a^\lambda.$$

Then, equality (5.4.29) imposes the following identification:

$$R^\lambda_{\rho\mu\nu} = R^{ab}{}_{\mu\nu} e_a^\lambda \theta_{b\rho}$$

where e^a and θ^a are the gauge frame and coframe, respectively.

The previous expression can be used in order to reexpress the Bianchi identities (*cf.* eqs.(5.3.16)) for a torsionfree Cartan-Poincaré geometry in the familiar form as:

First Bianchi identity:

$$\begin{aligned}
\left[\Theta^{\mathfrak{h}}, \theta^{\mathfrak{p}}\right] &= 0 \\
\left[R^{ab} J_{ab}, \theta^c P_c\right] &= 0 \\
R^a{}_b \wedge \theta^b &= 0 \\
R^\lambda{}_{[\rho\mu\nu]} &= 0.
\end{aligned}$$

Second Bianchi identity:

$$\begin{aligned}
d\Theta^{\mathfrak{h}} &= \left[\Theta^{\mathfrak{h}}, \omega^{\mathfrak{h}}\right] \\
d\left(R^{ab} J_{ab}\right) &= \left[R^{ab} J_{ab}, \omega^{cd} J_{cd}\right] \\
dR^{ab} &= R^a{}_c \wedge \omega^{cb} - R^b{}_c \wedge \omega^{ca} \\
\nabla_{[\alpha} R^{\lambda\rho}{}_{\mu\nu]} &= 0.
\end{aligned}$$

Chapter 6

Cartan viewpoint on nonrelativistic manifolds

Επειδη το ειδεναι και το επιστασθαι συμβαινει πεϱι πασας τας μεθοδους, ων εισιν αϱχαι η αιτια η στικεια, εκ του ταυτα γνωϱιξειν: τοτε γαϱ οιομετα γιγνωσκειν εκαστον, οταν τα αιτια γωρισωμεν τα πϱωτα και τας αϱχας τας πρωτας και μεχρι των στοιχειων. Δηλον οτι και της περι φυσεος επιστημης πειϱατεον διοϱισασθαι πρωτον τα πεϱι τας αϱχας.

In all disciplines in which there is systematic knowledge of things with principles, causes, or elements, it arises from a grasp of those: we think we have knowledge of a thing when we have found its primary causes and principles, and followed it back to its elements. Clearly, then, systematic knowledge of nature must start with an attempt to settle questions about principles.

– Aristotle, *Physics* Book I, Chapter 1, 184a 10-21

As mentioned in the introduction of this manuscript, the ambition of the present Chapter is to reinterpret Newton-Cartan structures as "Cartan-Newton" geometries, *i.e.* Cartan geometries modeled on nonrelativistic groups (Galilei or its central extension). Chapter 4 already imparted some naturality to nonrelativistic structures by reviewing their embedding inside gravitational waves. The present Chapter aims to forge ahead by showing how these nonrelativistic structures can be derived from first principles using Cartan's approach to differential geometry, as reviewed in Chapter 5. In this context, nonrelativistic geometries appear as natural as their relativistic cousins, differing only by the choice of structural Klein geometries (or model space). This Chapter should be primarily understood as preliminary work *i.e.* we wish here to make the case that Cartan geometries constitute a useful formalism regarding the description of nonrelativistic theories of gravitation (as well as their embedding inside relativistic manifolds) by reinterpreting some classic results in this formalism and hence, hopefully, pave the way to subsequent generalisations.

6.1 Galilean manifolds as Cartan-Galilei geometries

As a gentle start, we present how a Cartan-Galilei geometry (*i.e.* a Cartan geometry modeled on the pair $(\mathrm{Gal}, \mathrm{Gal}_0)$ where Gal designates the Galilei group while Gal_0 stands for the homogeneous Galilei group) induces on its base space $\mathscr{M}$ a structure of Galilean manifold.

Let (P, ω) be a Cartan geometry modeled on the infinitesimal Klein pair $(\mathfrak{gal}, \mathfrak{gal}_o)$ where $\mathfrak{gal}$ designates the Galilei algebra (*cf.* commutation relations (B.2.6)-(B.2.9)) and $\mathfrak{gal}_o$ is the homogeneous Lie subalgebra. Being reductive, the Lie algebra $\mathfrak{gal}$ admits the $\mathrm{Ad}\,(\mathrm{Gal}_0)$-module decomposition as : $\mathfrak{gal} = \mathfrak{gal}_o \oplus \mathfrak{p}$. We choose a basis for $\mathfrak{gal}$ such that $\mathfrak{gal}_o = \mathrm{Span}\,\{K_i, J_{ij}\}$ and $\mathfrak{p} = \mathrm{Span}\,\{H, P_i\}$ while the dual canonical basis decomposes as ${\mathfrak{gal}_o}^* = \mathrm{Span}\left\{K^{i*}, J^{ij*}\right\}$ and $\mathfrak{p}^* = \mathrm{Span}\left\{H^*, P^{i*}\right\}$. The principal Gal_0-bundle P over the manifold $\mathscr{M}$

$$\begin{tikzcd} \mathrm{Gal}_0 \arrow[d, hook] \\ P \arrow[d, "\pi"] \\ \mathscr{M} \end{tikzcd} \tag{6.1.1}$$

is endowed with a Cartan connection ω taking values in $\mathfrak{gal}$.
Our aim is now to make use of the Cartan setup in order to lower natural objects living on the Klein model space $\mathrm{Gal}/\mathrm{Gal}_0$ downto nonrelativistic structures living on the base manifold $\mathscr{M}$. The previous Section emphasised the naturalness of $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant structures in this type of construction, so that one starts by investigating $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\,(\mathrm{Gal}_0)$-invariant elements of $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$. Representation theory of the Galilei group (*cf.* Section B.2.3) singles out the two following elements

- $H^* \in \mathfrak{p}^*$
- $(\cdot, \cdot)_{\mathfrak{p}}^{-1} \equiv \delta^{ij} P_i \vee P_j \in \vee^2 \mathfrak{p}$

as the only $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant elements of $\bigotimes^p \mathfrak{p} \otimes \bigotimes^q \mathfrak{p}^*$, with $p + q \leq 2$.

Some comments are in order: the contravariant bilinear form $(\cdot, \cdot)_{\mathfrak{p}}^{-1}$ is seen to be degenerate with radical spanned by the linear form H^*. One is then dealing with a degenerate metric structure on $\mathfrak{p}$, to be contrasted with the full rank bilinear form emerging from the representation theory of the Poincaré group (*cf.* Example 5.1.7). The necessity to consider degenerate metric structures in nonrelativistic physics can then be traced back to the lack of $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(\mathrm{Gal}_0\right)$-invariant bilinear form on $\mathfrak{p}$ or its dual. However, applying Proposition A.1.5, one can forge a "spacelike" bilinear form acting on Span P_i = Ker H^* which is non-degenerate. This $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant bilinear form reads explicitly: $(\cdot,\cdot)_{\{P_i\}} \equiv \delta_{ij} P^{i*} \vee P^{j*} \in \vee^2 \,\mathrm{Span}\, P^{i*}$.

According to Proposition A.6.10, the $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(\mathrm{Gal}_0\right)$-invariant $\mathfrak{p}$-metric structure $\left(H^*, (\cdot,\cdot)^{-1}_{\mathfrak{p}}\right)$ defines tensors on P of the type $\mathscr{T}\left(\mathfrak{p}^*, \tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$ and $\mathscr{T}\left(\vee^2 \mathfrak{p}, \tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\right)$, respectively, which in turn can be lowered downto $\mathscr{M}$ using the isomorphism $\Psi^{-1} : \mathscr{T}(V,\rho) \to \Gamma(E)$ (*cf.* Definition 5.2.13 as well as Table 5.3). Explicitly, one constructs the 1-form $\psi \in \Omega^1(\mathscr{M})$ as $\psi \equiv \bar{\Psi}^{-1}(H^*)$ and the contravariant metric $h \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ as $h \equiv \tilde{\Psi}^{-1}\left((\cdot,\cdot)^{-1}_{\mathfrak{p}}\right)$. The fact that $\bar{\Psi}$ is an isomorphism guarantees that ψ never vanishes while Rad $\left((\cdot,\cdot)^{-1}_{\mathfrak{p}}\right)$ = Span H^* ensures that Rad (h_x) = Span ψ_x at each point $x \in \mathscr{M}$. We sum up this Section with the Proposition:

Proposition 6.1.1. *A Cartan-Galilei geometry induces a Leibnizian structure $\mathscr{L}(\mathscr{M}, \psi, h)$ whose degenerate metric structure (ψ, h) on the base space $\mathscr{M}$ is the natural counterpart of the metric structure $\left(H^*, (\cdot,\cdot)^{-1}_{\mathfrak{p}}\right)$ on the Klein-Galilei geometry.*

This Proposition can be seen as the analogue in the Cartan formalism of Theorem 3 in [18] where Leibnizian structures were described as G-structures for the homogeneous Galilei group Gal_0.

We now show how the canonical basis for the "translation" module $\mathfrak{p}$ = Span $\{H, P_i\}$ induces a (non-canonical) Galilean basis (*cf.* Definition 3.2.13 and Proposition 3.2.14) on the base space $\mathscr{M}$. The proposed construction makes use of the isomorphism $\varphi_p : T_x\mathscr{M} \to \mathfrak{p}$ (with $\pi(p) = x$) defined in Proposition 5.2.9 which precisely establishes a bijective correspondence between the vector spaces $\mathfrak{p}$ and $T_x\mathscr{M}$ and so can be used in order to import the canonical basis of $\mathfrak{p}$ downto the tangent space of $\mathscr{M}$ at x.

Proposition 6.1.2. *The tangent space at each point $x \in \mathscr{M}$ of the base space is endowed with a non-canonical Galilean basis imported from the canonical basis for the module $\mathfrak{p}$.*

Proof: Our aim is to show how the isomorphism $\varphi_p : T_x\mathscr{M} \to \mathfrak{p}$ (where $p \in \pi^{-1}(x)$ is arbitrary) maps the canonical basis $\{H, P_i\}$ onto a Galilean basis of the tangent space at $x \in \mathscr{M}$, which is non-canonical due to the arbitrariness of p. One is then led to define the vector $N^p_x \in T_x\mathscr{M}$ as $N^p_x \equiv \varphi_p^{-1}(H)$ as well as the set of d vectors $e^p_{i_x} \in T_x\mathscr{M}$, with $i = 1, \ldots, d$ as $e^p_{i_x} \equiv \varphi_p^{-1}(P_i)$. That fact that φ_p is an isomorphism ensures that $\left\{N^p_x, e^p_{i_x}\right\}$ is indeed a basis of $T_x\mathscr{M}$.

Now, in order for this induced basis to be Galilean, it has to satisfy Conditions 1-3 from Definition 3.2.13. That this is the case is easily shown, recalling the pointwise definition of the absolute clock and rulers, in the Cartanian setup, as $\psi_x \equiv \bar{\varphi}_p^{-1}(H^*)$ and $\gamma_x \equiv \bar{\bar{\varphi}}_p^{-1}\left(\delta_{ij} P^{i*} \vee P^{j*}\right)$, respectively. Conditions 1-3 are then seen to be satisfied as follows:

1. $\psi_x(N_x) = \psi_x\left(\varphi_p^{-1}(H)\right) = \left(\bar{\varphi}_p(\psi_x)\right)(H) = H^*(H) = 1.$
2. $\psi_x(e_{i_x}) = \psi_x\left(\varphi_p^{-1}(P_i)\right) = \left(\bar{\varphi}_p(\psi_x)\right)(P_i) = H^*(P_i) = 0.$
3. $\gamma_x(e_{i_x}, e_{j_x}) = \gamma_x\left(\varphi_p^{-1}(P_i), \varphi_p^{-1}(P_j)\right) = \left(\bar{\bar{\varphi}}_p \gamma_x\right)(P_i, P_j) = \left(\delta_{kl} P^{k*} \vee P^{l*}\right)(P_i, P_j) = \delta_{ij}.$

□

As emphasised in the previous Proposition, the basis $B_x^p \equiv \left\{N_x^p, e_{i_x}^p\right\}$ is not canonically defined in the sense that it depends on the choice of representative point $p \in \pi^{-1}(x)$, hence the superscript p. Switching the representative from p to $ph \equiv R_h p$, with

$$h \equiv \begin{pmatrix} 1 & \mathbf{b} \\ 0 & \mathbf{R} \end{pmatrix} \in \mathrm{Gal}_0, \tag{6.1.2}$$

where $\mathbf{b} \in \mathbb{R}^d$ and $\mathbf{R} \in O(d)$, leads yet to a different Galilean basis spanned by the vectors:

$$\begin{cases} N_x^{ph} \equiv \varphi_{ph}^{-1}(H) = \varphi_p^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h) H\right) = \varphi_p^{-1}\left(H + \mathbf{b}^i P_i\right) = N_x^p + \mathbf{b}^i e_{i_x}^p \\ e_{i_x}^{ph} \equiv \varphi_{ph}^{-1}(P_i) = \varphi_p^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h) P_i\right) = \varphi_p^{-1}\left(\mathbf{R}^j{}_i P_j\right) = \mathbf{R}^j{}_i e_{j_x}^p. \end{cases} \tag{6.1.3}$$

One recognises the group action (3.2.14) of the homogeneous Galilei group Gal_0 on the space of Galilean bases. Furthermore, the regularity of the right-action on each fiber $\pi^{-1}(x)$ of the principal bundle P (or equivalently the regularity of $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : \mathrm{Gal}_0 \to \mathrm{End}(\mathfrak{p})$) ensures that the homogeneous Galilei group acts regularly on the space of Galilean bases, See Proposition 3.2.14.

From expressions (6.1.3), one concludes that the non-canonical character of the basis $B_x^p \equiv \left\{N_x^p, e_{i_x}^p\right\}$ can be traced back to the non-invariance of the basis $\{H, P_i\}$ under the adjoint action $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$. This intimate relationship between $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariance and canonicality will be a recurrent feature of our present approach.

The previous setup can be easily applied to the construction of a (again, non-canonical) basis of $T_x^*\mathscr{M}$ starting from the canonical basis $\left\{H^*, P^{i*}\right\}$ and making use of the isomorphism $\bar{\varphi}_p : T_x^*\mathscr{M} \to \mathfrak{p}^*$ (*cf.* Proposition 5.2.10).

The base counterpart of H^* is, unsurprisingly, the absolute clock ψ_x whose canonicality is ensured by the $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}$-invariance of H^* while the d linear forms P^{i*} define a set of d

1-forms on $T_x^*\mathscr{M}$ as $\theta_x^{ip} \equiv \bar{\varphi}_p^{-1}\left(P^{i*}\right)$. It is then easy to check that the basis $B_x^* = \left\{\psi_x, \theta_x^{ip}\right\}$ thus constructed is dual to the basis $B_x^p \equiv \left\{N_x^p, e^p_{i_x}\right\}$, since, in addition to the relations $\psi_x\left(N_x^p\right) = 1$, $\psi_x\left(e^p_{i_x}\right) = 0$, one gets:

$$\begin{cases} \theta_x^{ip}\left(N_x^p\right) = \bar{\varphi}_p^{-1}\left(P^{i*}\right)\left(N_x^p\right) = P^{i*}\left(\varphi_p\left(N_x^p\right)\right) = P^{i*}\left(H\right) = 0 \\ \theta_x^{ip}\left(e^p_{j_x}\right) = \bar{\varphi}_p^{-1}\left(P^{i*}\right)\left(e^p_{j_x}\right) = P^{i*}\left(\varphi_p\left(e^p_{j_x}\right)\right) = P^{i*}\left(P_j\right) = \delta^i_j. \end{cases} \tag{6.1.4}$$

It will again be instructive to know how this basis transforms under a change of representative $p \to ph$:

$$\begin{cases} \psi_x \equiv \bar{\varphi}_{ph}^{-1}\left(H^*\right) = \bar{\varphi}_p^{-1}\left(\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)H^*\right) = \bar{\varphi}_p^{-1}\left(H^*\right) = \psi_x \\ \theta_x^{ip} \equiv \bar{\varphi}_{ph}^{-1}\left(P^{i*}\right) = \bar{\varphi}_p^{-1}\left(\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)P^{i*}\right) = \varphi_p^{-1}\left(-\mathbf{R}^{T}{}^i{}_j\mathbf{b}^jH^* + \mathbf{R}^{T}{}^i{}_jP^{j*}\right) = -\mathbf{R}^{T}{}^i{}_j\mathbf{b}^j\psi_x + \mathbf{R}^{T}{}^i{}_j\theta_x^{jp}. \end{cases}$$

The generalisation from the pointwise case to the field case necessitates a prescription to assign to each point $x \in \mathscr{M}$ a specific point $p \in \pi^{-1}\left(x\right)$ *i.e.* a choice of section $\sigma : \mathscr{U} \subset \mathscr{M} \to P$, where $\mathscr{U}$ is an open set of $\mathscr{M}$ (*cf.* Example 5.2.16). Let $\overset{\sigma}{\Upsilon}{}^{-1} : C^\infty\left(\mathscr{U}\right) \otimes \mathfrak{p} \to \Gamma\left(T\mathscr{U}\right)$ designate the (section dependent) map between functions on $\mathscr{U}$ taking values in $\mathfrak{p}$ and vector fields on $\mathscr{U}$ defined as $\overset{\sigma}{\Upsilon}{}^{-1}\left(\phi\right)\left(x\right) = \varphi^{-1}_{\sigma(x)}\left(\phi\left(x\right)\right)$, with $\phi \in C^\infty\left(\mathscr{U}\right) \otimes \mathfrak{p}$. Let us emphasise that the vector field $\overset{\sigma}{\Upsilon}{}^{-1}\left(\phi\right) \in \Gamma\left(T\mathscr{U}\right)$ generically depends on the section σ and transforms under a change of section $\sigma \Rightarrow_h \sigma'$ parameterised by the map $h : \mathscr{U} \cap \mathscr{V} \to \mathrm{Gal}_0$ as: $\overset{\sigma'}{\Upsilon}{}^{-1}\left(\phi\right) = \overset{\sigma}{\Upsilon}{}^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)\phi\right)$. The map $\overset{\sigma}{\Upsilon}{}^{-1}$ can be generalised in the usual way in order to map functions of $\mathscr{U}$ taking values in $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$ and fields on $\mathscr{U}$. Equipped with this map, one defines the vector fields $\overset{\sigma}{N}$ and $\overset{\sigma}{e}_i \in \Gamma\left(T\mathscr{U}\right)$ as $\overset{\sigma}{N} \equiv \overset{\sigma}{\Upsilon}{}^{-1}\left(H\right)$ and $\overset{\sigma}{e}_i \equiv \overset{\sigma}{\Upsilon}{}^{-1}\left(P_i\right)$. The canonical basis $\left\{H, P_i\right\}$ of $\mathfrak{p}$ is then mapped into a non-canonical (*i.e.* section dependent) Galilean basis $\{\overset{\sigma}{N}, \overset{\sigma}{e}_i\}$ of $\Gamma\left(T\mathscr{U}\right)$. A basis of $\Omega^1\left(\mathscr{U}\right)$ can be constructed in a similar fashion, making use of the map $\overset{\sigma}{\bar{\Upsilon}}{}^{-1} : C^\infty\left(\mathscr{U}\right) \otimes \mathfrak{p}^* \to \Omega^1\left(\mathscr{U}\right)$ defined as $\overset{\sigma}{\bar{\Upsilon}}{}^{-1}\left(\alpha\right)\left(X\right) = \alpha\left(\overset{\sigma}{\Upsilon}\left(X\right)\right)$, with $\alpha : \mathscr{U} \to \mathfrak{p}^*$ and $X \in \Gamma\left(T\mathscr{U}\right)$. The 1-form $\overset{\sigma}{\bar{\Upsilon}}{}^{-1}\left(H^*\right)$ is invariant under a change of section, due to the $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}$ (Gal)-invariance of H^*, and coincides then with the absolute clock ψ. As for the spacelike coframe 1-form, it is defined as $\overset{\sigma}{\theta}{}^i \equiv \overset{\sigma}{\bar{\Upsilon}}{}^{-1}\left(P^{i*}\right)$ so that the non-canonical basis for $\Omega^1\left(\mathscr{U}\right)$ reads $\{\psi, \overset{\sigma}{\theta}{}^i\}$. This basis is by construction the dual basis to $\{\overset{\sigma}{N}, \overset{\sigma}{e}_i\}$. Given an arbitrary vector field $X \in \Gamma\left(T\mathscr{U}\right)$, the following consistency relation must be satisfied:

$$\overset{\sigma}{\Upsilon}{}^{-1}\left(\overset{\sigma}{\Upsilon}\left(X\right)\right) = X. \tag{6.1.5}$$

The left-hand side can be expressed as follows:

$$\overset{\sigma}{\Upsilon}^{-1}\left(\overset{\sigma}{\Upsilon}(X)\right) = \overset{\sigma}{\Upsilon}^{-1}\left(\psi(X)\,H+\overset{\sigma}{\theta}^{i}(X)\,P_i\right) \tag{6.1.6}$$

$$= \left(\overset{\sigma}{N}H^*+\overset{\sigma}{e}_jP^{j*}\right)\left(\psi(X)\,H+\overset{\sigma}{\theta}^{i}(X)\,P_i\right) \tag{6.1.7}$$

$$= \psi(X)\,\overset{\sigma}{N}+\overset{\sigma}{\theta}^{i}(X)\,\overset{\sigma}{e}_i. \tag{6.1.8}$$

Requiring $\overset{\sigma}{\Upsilon}^{-1}\left(\overset{\sigma}{\Upsilon}(X)\right)=X$ then imposes $\psi(X)\,\overset{\sigma}{N}+\overset{\sigma}{\theta}^{i}(X)\,\overset{\sigma}{e}_i=X,\ \forall X\in\Gamma(T\mathscr{U})$. Expressing this condition in components together with the duality between spacelike frame and coframe leads to:

$$\begin{cases}\overset{\sigma}{\theta}{}^{i}_{\mu}\,\overset{\sigma}{e}{}^{\mu}_{j}=\delta^{i}_{j}\\ \overset{\sigma}{\theta}{}^{i}_{\nu}\,\overset{\sigma}{e}{}^{\mu}_{i}=\delta^{\mu}_{\nu}-\overset{\sigma}{N}{}^{\mu}\psi_\nu.\end{cases} \tag{6.1.9}$$

The second expression illustrates the degenerate character of the Galilei frame (compare with the Poincaré case in eq. (5.4.23)).

Under a change of section parameterised by the map $h:\mathscr{U}\cap\mathscr{V}\to\mathrm{Gal}_0$ defined as

$$h=\begin{pmatrix}1&\mathbf{b}\\0&\mathbf{R}\end{pmatrix} \tag{6.1.10}$$

where $\mathbf{R}:\mathscr{U}\cap\mathscr{V}\to O(d)$ parameterise the local rotations and $\mathbf{b}^i:\mathscr{U}\cap\mathscr{V}\to\mathbb{R}^d$ the local Galilean boosts, the vector field $\overset{\sigma}{N}$ transforms as $\overset{\sigma'}{N}=\overset{\sigma}{\Upsilon}^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,H\right)=\overset{\sigma}{\Upsilon}^{-1}\left(H+\mathbf{b}^iP_i\right)=\overset{\sigma}{N}+\mathbf{b}^i\overset{\sigma}{e}_i$ while the spacelike frame $\overset{\sigma}{e}_i$ varies according to $\overset{\sigma'}{e}_i=\overset{\sigma}{\Upsilon}^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,P_i\right)=\overset{\sigma}{\Upsilon}^{-1}\left(\mathbf{R}^j{}_i\,P_j\right)=\mathbf{R}^j{}_i\,\overset{\sigma}{e}_j$. The spacelike frame field $\overset{\sigma}{e}_i$ can be used in order to express the contravariant bilinear form h as $h\equiv\tilde{\Psi}^{-1}\left(\delta^{ij}P_i\vee P_j\right)=\overset{\sigma}{\Upsilon}^{-1}\left(\delta^{ij}P_i\vee P_j\right)=\delta^{ij}\overset{\sigma}{e}_i\overset{\sigma}{e}_j$ which is indeed invariant under a change of section due to the orthogonality of $\mathbf{R}$.

This expression for the Leibnizian metric h is useful in order to reformulate the transformation law of $\overset{\sigma}{N}$ under a change of section as $\overset{\sigma'}{N}=\overset{\sigma}{N}+\mathbf{b}^i\overset{\sigma}{e}_i=\overset{\sigma}{N}+h^{\mu\nu}\overset{\sigma}{\theta}_{i\nu}\mathbf{b}^i=\overset{\sigma}{N}+h(\chi)$, where the 1-form $\chi\in\Omega^1(\mathscr{U}\cap\mathscr{V})$ is defined[1] as $\chi\equiv\overset{\sigma}{\theta}_i\mathbf{b}^i$. A change of section in the principal bundle $\mathrm{Gal}_0\hookrightarrow P\longrightarrow\mathscr{U}\cap\mathscr{V}$ parameterised by the local Galilean boost

1. Remember that the 1-form χ can always be chosen to be "spacelike" in the sense that $\chi=\bar{P}^N(\chi)$ for some field of observers N (*cf.* the discussion following Proposition 3.2.10).

$\mathbf{b}^i : \mathscr{U} \cap \mathscr{V} \to \mathbb{R}^d$ takes then the nonrelativistic interpretation of a Milne boost parameterised by the 1-form $\chi \equiv \overset{\sigma}{\theta}_i \mathbf{b}^i$.

The spacelike coframe 1-form $\overset{\sigma}{\theta}{}^i \equiv \overset{\sigma}{\Upsilon}{}^{-1}\left(P^{i*}\right)$ transforms as $\overset{\sigma'}{\theta}{}^i = \overset{\sigma}{\Upsilon}{}^{-1}\left(-\mathbf{R}^{T\,i}{}_j \mathbf{b}^j H^* + \mathbf{R}^{T\,i}{}_j P^{j*}\right) = -\mathbf{R}^{T\,i}{}_j \mathbf{b}^j \psi + \mathbf{R}^{T\,i}{}_j \overset{\sigma}{\theta}{}^i$. The spacelike coframe can be used in order to define the covariant bilinear form $\overset{\sigma}{\gamma} \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ as $\overset{\sigma}{\gamma} \equiv \overset{\sigma}{\theta}{}^i \overset{\sigma}{\theta}{}^j \delta_{ij}$, which is nothing but the base version of the bilinear form $\delta_{ij} P^{i*} \vee P^{j*} \in \mathfrak{p}^* \vee \mathfrak{p}^*$, *i.e.* $\overset{\sigma}{\gamma} \equiv \overset{\sigma}{\bar{\bar{\Upsilon}}}{}^{-1}\left(\delta_{ij} P^{i*} \vee P^{j*}\right)$. In nonrelativistic physics, the bilinear form $\overset{\sigma}{\gamma}$ takes the interpretation of the transverse metric to $\overset{\sigma}{N}$ (*cf.* Definition 3.2.11). Under a gauge transformation, $\overset{\sigma}{\gamma}$ is modified as:

$$
\begin{aligned}
\overset{\sigma'}{\gamma} &= \overset{\sigma'}{\Upsilon}{}^{-1}\left(\delta_{ij} P^{i*} \vee P^{j*}\right) \\
&= \overset{\sigma}{\Upsilon}{}^{-1}\left(\bar{\bar{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}(h)\left(\delta_{ij} P^{i*} \vee P^{j*}\right)\right) \\
&= \overset{\sigma}{\Upsilon}{}^{-1}\left(\delta_{ij} P^{i*} \vee P^{j*} - 2\delta_{ij} \mathbf{b}^i H^* \vee P^{j*} + \delta_{ij} \mathbf{b}^i \mathbf{b}^j H^* \vee H^*\right) \\
&= \delta_{ij} \overset{\sigma}{\theta}{}^i \overset{\sigma}{\theta}{}^j - 2\mathbf{b}_i^T \psi \vee \overset{\sigma}{\theta}{}^i + \mathbf{b}^2 \psi \vee \psi \\
&= \overset{\sigma}{\gamma} - 2\chi \vee \psi + h(\chi,\chi)\, \psi \vee \psi
\end{aligned}
$$

which is seen to match expression (3.2.10) of the transformation of the transverse metric under a Milne boost parameterised by the 1-form $\chi \equiv \overset{\sigma}{\theta}_i \mathbf{b}^i$.

All the algebraic (in the sense of non-differential *i.e.* without notion of parallelism) concepts and relations of standard nonrelativistic physics are then seen to be recast in the Cartanian formulation as arising naturally from group-theoretical considerations. The principal objects discussed previously are summed up in the following table:

$\mathfrak{p}$	$\Gamma\left(T\mathscr{M}\right)$	$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariance	$\mathfrak{p}^*$	$\Omega^1\left(\mathscr{M}\right)$	$\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}$-invariance
H	$\overset{\sigma}{N}$	$\times$	H^*	ψ	✓
P_i	$\overset{\sigma}{e}_i$	$\times$	P^{i*}	$\overset{\sigma}{\theta}{}^i$	$\times$
$\vee^2\,\mathfrak{p}$	$\Gamma\left(\vee^2 T\mathscr{M}\right)$	$\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}$-invariance	$\vee^2\,\mathfrak{p}^*$	$\Gamma\left(\vee^2 T^*\mathscr{M}\right)$	$\bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}$-invariance
$\delta^{ij}P_i\vee P_j$	h	✓	$\delta_{ij}P^{i*}\vee P^{j*}$	$\overset{\sigma}{\gamma}$	$\times$
$\vee^2\,\mathrm{Span}\ P_i$	$\Gamma\left(\vee^2\,\mathrm{Ker}\ \psi\right)$	$\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}$-invariance	$\vee^2\,\mathrm{Span}\ P^{i*}$	$\Gamma\left(\vee^2\ (\mathrm{Ker}\ \psi)^*\right)$	$\bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}$-invariance
$\delta^{ij}P_i\vee P_j$	γ^{-1}	✓	$\delta_{ij}P^{i*}\vee P^{j*}$	γ	✓

Progressing towards richer structures, we now investigate how a notion of parallelism comes into play in this setup.

According to Propositions 5.3.13 and 5.3.14, a reductive Cartan-Galilei geometry possesses a well-defined Koszul connection $\nabla_X : \Gamma\left(E\right) \to \Gamma\left(E\right)$ with $X \in \Gamma\left(T\mathscr{M}\right)$ and where $E = P \times_{\mathrm{Gal}_0} (V, \rho)$ is the vector bundle associated to P whose sections $f \in \Gamma\left(E\right)$ are in one-to-one correspondence with tensors of type $\mathscr{T}\left(V, \rho\right)$ via the isomorphism Ψ (*cf.* Section A.6). Explicitly, ∇_X is defined by $\Psi\left(\nabla_X f\right) = \tilde{X}\left[\Psi\left(f\right)\right]$ with $\tilde{X}$ the horizontal lift of X and $f \in \Gamma\left(E\right)$.

The base manifold $\mathscr{M}$ is then endowed with a notion of parallelism, in the guise of the Cartan Koszul connection ∇. The natural question arises to determine which objects among the ones previously defined are preserved by ∇. This problem is in fact made trivial by making use of Proposition 5.3.15 which asserts that objects built from constant $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant elements of $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$ are parallelised by ∇. In particular, the absolute clock ψ and rulers γ (or h) belonging to the Leibnizian structure of Proposition 6.1.1 are deductibly seen to be preserved by ∇.

The following Proposition follows straightforwardly:

Proposition 6.1.3. *The base space of a Cartan-Galilei geometry is a Galilean manifold.*

By mirroring Example 5.4.11, one is able to obtain an expression for the Cartan Galilei Koszul connection. We start by decomposing the part of the Cartan Galilei gauge connection (in the gauge (θ, U)) taking values in the homogeneous Galilei algebra $\mathfrak{gal}_o = \mathrm{Span}\{K_i, J_{ij}\}$ as $\theta^{\mathfrak{h}} \equiv \omega^i K_i + \omega^{ij} J_{ij}$, where $\omega^i \in \Omega^1(U)$ is a set of d 1-form fields on U while $\omega^{ij} \in \Omega^1(U)$ are $\frac{d(d-1)}{2}$ 1-form fields on U antisymmmetric in (i,j). These two sets of 1-forms will be referred to as temporal and spatial spin-connections, respectively. Using the gauge expression for the Cartan Koszul connection (5.4.24) (with normalisation $\rho_* = -\frac{i}{2}\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}$), one obtains the following decomposition on the Galilean basis associated to the chosen gauge (dropping the superscript U for the sake of readibility):

$$\nabla_X Y = X\left[\psi(Y)\right] N + \left(X\left[\theta^i(Y)\right] + \frac{1}{2}\omega^i(X)\,\psi(Y) + \omega^{ij}(X)\,\theta_j(Y)\right) e_i$$

where X, Y are arbitrary vector fields. Making contact with the expression of the Koszul connection in an holonomic basis $\nabla_X Y \equiv X^\mu\left(\partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu} Y^\nu\right)\partial_\lambda$ leads to the Galilei version of the "first vielbein postulate":

$$\Gamma^\lambda_{\mu\nu} = N^\lambda \partial_\mu \psi_\nu + e^\lambda_i\left(\partial_\mu \theta^i_\nu + \omega^i_{\mu\ j}\theta^j_\nu + \frac{1}{2}\omega^i_\mu \psi_\nu\right). \tag{6.1.11}$$

We now focus on the torsionfree case *i.e.* when the Christoffel symbols (6.1.11) are symmetric, yielding the two constraints:

$$\begin{aligned}
&\partial_{[\mu}\psi_{\nu]} = 0\\
&\partial_{[\mu}\theta^i_{\nu]} + \omega^i{}_{j\,[\mu}\theta^j_{\nu]} + \frac{1}{2}\,\omega^i_{[\mu}\psi_{\nu]} = 0.
\end{aligned}$$

The first torsionfree constraint imposes that the absolute clock ψ is closed. This was expected in light of Proposition 3.2.20 which implies that only Augustinian structures (*cf.* Definition 3.2.16) can be supplemented with a compatible torsionfree Koszul connection. The second constraint can be solved in order to express the spatial spin-connection ω^{ij} in terms of the frames and the temporal spin-connection as:

$$\omega^{ij}_\mu = e^{\nu[i}\,\partial_{[\mu}\theta^{j]}_{\nu]} - e^{\rho[i}\,e^{\sigma j]}\,\partial_\rho\theta^k_\sigma\,\theta_{k\mu} - \frac{1}{2}\psi_\mu e^{\nu[i}\omega^{j]}_\nu \tag{6.1.12}$$

Following [86], we can express the temporal spin-connection as

$$\omega_{k\rho} = \psi_\rho N^\mu e^\nu_k \overset{N}{F}_{\mu\nu} + e^\nu_k \overset{N}{F}_{\rho\nu} - 2\theta^i_\rho N^\lambda \partial_{[\lambda}\theta_{\gamma]i} e^\gamma_k + 2N^\lambda \partial_{[\rho}\theta_{\lambda]k} \tag{6.1.13}$$

where the 2-form $\overset{N}{F} \in \Omega^2(\mathscr{M})$ defined as $\overset{N}{F}_{\mu\nu} = \omega^i_{[\mu}\theta_{\nu]i}$ encodes the part of the temporal spin-connection that cannot be resolved in terms of the frames and coframes using the torsionfree conditions. This is the Cartanian analogue of the non-uniqueness of torsionfree Galilean connections preserving a given Augustinian structure.

Plugging back expressions (6.1.12) and (6.1.13) into the first vielbein postulate (6.1.11) gives the coefficients (3.2.20) for a torsionfree Galilean connection:

$$\Gamma^{\lambda}_{\mu\nu} = N^{\lambda}\partial_{(\mu}\psi_{\nu)} + \frac{1}{2}h^{\lambda\rho}\left[\partial_{\mu}\overset{N}{\gamma}_{\rho\nu} + \partial_{\nu}\overset{N}{\gamma}_{\rho\mu} - \partial_{\rho}\overset{N}{\gamma}_{\mu\nu}\right] + h^{\lambda\rho}\psi_{(\mu}\overset{N}{F}_{\nu)\rho}. \tag{6.1.14}$$

We conclude this Section by displaying the relation between the Galilei gauge curvature and the Riemann tensor for ∇. We first decompose the gauge curvature $\Theta^{\mathfrak{gal}_o} \in \Omega^2(\mathscr{M}) \otimes \mathfrak{gal}_o$ on the basis $\mathfrak{gal}_o = \text{Span}\{K_i, J_{ij}\}$ as $\Theta^{\mathfrak{gal}_o} \equiv R^{ij}J_{ij} + R^iK_i$. Now, eq.(5.4.29) ensures that

$$R^{\lambda}_{\rho\mu\nu} = R^{ij}{}_{\mu\nu}\theta_{j\rho}e^{\lambda}_{i} - R^{i}{}_{\mu\nu}\psi_{\rho}e^{\lambda}_{i} \tag{6.1.15}$$

where $R^{\lambda}_{\rho\mu\nu}$ are the components of the Riemann tensor associated to the Cartan Koszul connection ∇ in holonomic coordinates.

At this point, one may wonder if there is a way to proceed further by making our base space a Newtonian manifold. The answer is yes but one needs additional Cartan-structure in order to manage it. Namely, one needs to embed the Cartan-Galilei geometry into a Cartan-Bargmann geometry. This will provide us with the additional structure necessary to impose the Duval-Künzle condition (*cf.* Definition 3.2.28) and thus to define a Newtonian manifold.

6.2 Cartan-Bargmann geometry and the Duval-Künzle condition

Newtonian manifolds have been defined in Section 3.2 as Galilean manifolds satisfying an extra condition on the curvature of the Koszul connection, the Duval-Künzle condition 3.2.28. The naturalness of this condition has been emphasised in Section 4.4.1 in the ambient framework by reviewing the work [24] in which this condition was shown to originate from a (trivial) symmetry relation enjoyed by the Riemann tensor associated to the Levi-Civita of the embedding Bargmann-Eisenhart wave. In the present Section, we aim at providing a purely nonrelativistic justification for the Duval-Künzle condition in the Cartan formalism, by embedding a Cartan-Galilei geometry inside a Cartan-Bargmann geometry. It is indeed hard to overstate the relevance of the Bargmann group (the only non-trivial central extension of the Galilei group (in dimension $d \geq 4$)) when the geometrising of nonrelativistic physics is concerned, as has been emphasised in the seminal works [53, 152, 24]. Notably, in the work [53], Newtonian connections have been characterised using the Bargmann algebra, in the formalism of affine connections (in the sense of §III.3 of [153]). Our aim in this Section is to reinterpret this important result in the context of Cartan geometries, in which the Duval-Künzle condition will be seen to naturally emerge as an involutivity condition, providing a slightly different take on its purely nonrelativistic interpretation.

Before introducing the geometrical setup, let us briefly display the main group-theoretical protagonists at hand in the present construction. The Bargmann algebra $\mathfrak{bar}$ can be expressed as the semi-direct sum $\mathfrak{bar} = (\mathfrak{o}\,(d) \oplus \mathbb{R}) \in \mathfrak{h}^d$ where $\mathfrak{h}^d$ stands for the Heisenberg algebra $\mathfrak{h}^d = \text{Span}\,\{P_i, K_i, M\}$, with non-trivial commutation relation $[P_i, K_j] = i\delta_{ij}M$. The Bargmann algebra is thus seen to supplement the Galilei algebra $\mathfrak{gal} = (\mathfrak{o}\,(d) \oplus \mathbb{R}) \in \mathbb{R}^{2d}$ with the "mass operator" M, belonging to the center of $\mathfrak{bar}$, *i.e.* $[M, X] = 0$, $\forall X \in \mathfrak{bar}$.

The (inhomogeneous) Bargmann group can be represented as the group of matrices (*cf.* [154, 24]):

$$\begin{pmatrix} 1 & f & e & \mathbf{c} \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}\mathbf{b}^T\mathbf{b} & 1 & \mathbf{b} \\ 0 & -\mathbf{b}^T\mathbf{R} & 0 & \mathbf{R} \end{pmatrix}$$

with

$$\begin{cases} \mathbf{R} \in O\,(d) : \text{Orthogonal transformations} \\ \mathbf{b} \in \mathbb{R}^d : \text{Galilean boosts} \\ \mathbf{c} \in \mathbb{R}^d : \text{Spatial translations} \\ e \in \mathbb{R} : \text{Temporal translations} \\ f \in \mathbb{R} : \text{Vertical translation.} \end{cases}$$

From this representation, one concludes that the stabiliser of a point is given by the direct product $\text{Gal}_0 \times \mathbb{R}$, where $\text{Gal}_0 \equiv O\,(d) \ltimes \mathbb{R}^d$ is the homogeneous Galilei group (*cf.* the matrix representation (3.2.13)) and $\mathbb{R}$ stands for the mass operator. At the algebra level, this induces a decomposition of the Bargmann Lie algebra, contemplated as a vector space, between "homogeneous" and space and time "translations" parts as $\mathfrak{bar} = (\mathfrak{gal}_o \oplus \mathbb{R}) \oplus \mathfrak{p}$, where $\mathfrak{gal}_o \oplus \mathbb{R} = \text{Span}\,\{J_{ij}, K_i, M\}$ is the Lie subalgebra associated to the homogeneous subgroup $\text{Gal}_0 \times \mathbb{R}$ while the vector space $\mathfrak{p}$ is spanned by the translation operators: $\mathfrak{p} = \text{Span}\,\{P_i, H\}$.

At this point, it is important to notice that this vector space decomposition is *not* reductive since the translation vector space $\mathfrak{p}$ is not an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Gal}_0 \times \mathbb{R})$-module. This fact is easily understood at the Lie algebra level, since $[P_i, K_j] \in \mathrm{Span}\, M \notin \mathfrak{p}$, so that the vector space $\mathfrak{p}$ is not stable by the adjoint action $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(\mathfrak{gal}_0 \oplus \mathbb{R})$. The geometric consequences of this fact will be appreciated soon.

We now define a Cartan-Bargmann geometry (P, ω) modeled on the infinitesimal Klein pair $(\mathfrak{bar}, \mathfrak{gal}_o \oplus \mathbb{R})$ with P a $(\mathrm{Gal}_0 \times \mathbb{R})$-bundle:

$$\begin{tikzcd} \mathrm{Gal}_0 \times \mathbb{R} \arrow[d, hook] \\ P \arrow[d, "\pi"] \\ \mathscr{M}. \end{tikzcd}$$

Let $\Omega \in \Omega^2(P) \otimes \mathfrak{bar}$ denote the curvature 2-form associated to the Cartan connection $\omega \in \Omega^1(P) \otimes \mathfrak{bar}$. We will make the assumption that the Cartan-Bargmann geometry (P, ω) is torsionfree (*i.e.* Ω takes values in the homogeneous Lie subalgebra $\mathfrak{gal}_o \oplus \mathbb{R} = \mathrm{Span}\{J_{ij}, K_i, M\}$) as is natural in order to yield a torsionfree Koszul connection on the base manifold $\mathscr{M}$ (*cf.* Proposition 5.3.18).

Comparing with the Cartan-Galilei geometry described by Diagram 6.1.1, it must be noted that the base manifold has same dimension in these two geometries. This is clear, considering that the extra mass operator has been added to the homogeneous part of the Lie algebra, so that the dimension of the quotient remains unchanged. Now, according to the previous discussion, one can conclude that the Cartan-Bargmann geometry thus defined is not reductive, since the Klein geometry $(\mathfrak{bar}, \mathfrak{gal}_o \oplus \mathbb{R})$ is not. An obvious consequence is the fact that the Cartan-Bargmann geometry (P, ω) does not induce a canonical Koszul connection on the base space $\mathscr{M}$. This is clearly dissatisfactory since our aim is to endow the base space $\mathscr{M}$ with a Newtonian connection inherited from the Cartan geometry. However, since the infinitesimal Klein pair $(\mathfrak{gal}, \mathfrak{gal}_o)$ is reductive (*cf.* Section 6.1), the possibility remains to perform a reduction of the structure group (*cf.* Definition A.8.6) from $\mathrm{Gal}_0 \times \mathbb{R}$ to Gal_0 in order to yield a reductive Cartan geometry, namely a Cartan-Galilei geometry. We thus suppose the existence of a ρ-imbedding $i : \bar{P} \to P$ with $\rho : \mathrm{Gal}_0 \to \mathrm{Gal}_0 \times \mathbb{R}$ the inclusion homorphism. The ρ-imbedding i is supposed to satisfy the conditions of Definition A.8.6, so that the subbundle $\bar{P}$ is a Gal_0-bundle on $\mathscr{M}$ with projection $\bar{\pi} : \bar{P} \to \mathscr{M}$ defined as $\bar{\pi} \equiv \pi \circ i$:

$$\begin{tikzcd}
\mathrm{Gal}_0 \arrow[r, hook, "\rho"] \arrow[d, hook] & \mathrm{Gal}_0 \times \mathbb{R} \arrow[d, hook] \\
\bar{P} \arrow[r, hook, "i"] \arrow[d, "\bar{\pi}"'] & P \arrow[dl, "\pi"] \\
\mathscr{M} &
\end{tikzcd} \tag{6.2.16}$$

In order for $\mathrm{Gal}_0 \hookrightarrow \bar{P} \to \mathscr{M}$ to be a Cartan-Galilei geometry, the subbunde $\bar{P}$ must be endowed with a Cartan-Galilei connection $\bar{\omega} \in \Omega^1\left(\bar{P}\right) \times \mathfrak{gal}$. A natural candidate for such a connection is the pullback of the Cartan-Bargmann connection $\omega \in \Omega^1(P) \times \mathfrak{bar}$ by the imbedding $i : \bar{P} \to P$, *i.e.* $\bar{\omega} \equiv i^* \omega$. Obviously, in order for $\bar{\omega}$ to take values in the Galilei algebra $\mathfrak{gal}$, consistency imposes that $\left(i^* \omega_{i(\bar{p})}\right)\left(\bar{X}_{\bar{p}}\right) \in \mathfrak{gal}\ \forall\, \bar{X}_{\bar{p}} \in T_{\bar{p}} \bar{P}$ so that

$$\omega_{i(\bar{p})}\left(i_* \bar{X}_{\bar{p}}\right) \in \mathfrak{gal},\ \forall\, \bar{X}_{\bar{p}} \in T_{\bar{p}} \bar{P}. \tag{6.2.17}$$

Defining the co-rank 1 distribution $\mathcal{D}$ (*cf.* Definition A.3.1) as the collection $\mathcal{D} = \{\mathcal{D}_p\}$ of co-rank 1 subspaces $\mathcal{D}_p \subset T_p P$ where

$$\mathcal{D}_p \equiv \omega_p^{-1}(\mathfrak{gal}) = \{X_p \in T_p P / \omega_p (X_p) \in \mathfrak{gal}\}, \tag{6.2.18}$$

condition (6.2.17) can be reformulated as $i_*\left(T_{(\bar{p})} \bar{P}\right) \in \mathcal{D}_{i(\bar{p})}$, $\forall \bar{p} \in \bar{P}$. Since $T\bar{P}$ and $\mathcal{D}_p$ have the same dimension and $i_* : T\bar{P} \to TP$ is injective, then

$$i_*\left(T_{(\bar{p})} \bar{P}\right) = \mathcal{D}_{i(\bar{p})},\ \forall \bar{p} \in \bar{P}$$

and $\bar{P}$ is thus an integral manifold for the distribution $\mathcal{D}$, *cf.* Definition A.3.3. The existence of such an integral manifold is guaranteed if the distribution $\mathcal{D}$ is integrable, hence involutive, by the means of Frobenius Theorem (*cf.* Theorem A.3.7).

The framework regarding the reduction of a Cartan geometry to an integral submanifold for a given involutive distribution is provided by the following Proposition:

Proposition 6.2.1. *Let (P, ω) be a Cartan geometry over $\mathscr{M}$ modeled on the infinitesimal Klein pair $(\mathfrak{g}, \mathfrak{h})$ admitting the vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. We let $\bar{P}$ be a reduced $\bar{H}$-subbundle of P with $\bar{H} \subset H$, so that the map $i : \bar{P} \hookrightarrow P$ is a ε-imbedding with $\varepsilon : \bar{H} \hookrightarrow H$*

an injective Lie group homomorphism:

$$\begin{tikzcd}
\bar{H} \arrow[r, hook, "\varepsilon"] \arrow[d, hook] & H \arrow[d, hook] \\
\bar{P} \arrow[r, hook, "i"] \arrow[d, "\bar{\pi}"'] & P \arrow[dl, "\pi"] \\
\mathscr{M} &
\end{tikzcd}$$

Let $\bar{\mathfrak{g}} \subset \mathfrak{g}$ be the Lie subalgebra of $\mathfrak{g}$ defined as the direct sum $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \mathfrak{p}$, where $\bar{\mathfrak{h}}$ is the Lie algebra of the subgroup $\bar{H}$. Let furthermore $\mathcal{D}$ be an involutive distribution on P defined as the collection $\mathcal{D} = \{\mathcal{D}_p\}$ where $\mathcal{D}_p \subset T_pP$ defined by $\mathcal{D}_p \equiv \omega_p^{-1}(\bar{\mathfrak{g}})$. Suppose that P is an integral manifold for the distribution $\mathcal{D}$. Then the 1-form $\bar{\omega} \in \Omega^1(\bar{P}) \otimes \bar{\mathfrak{g}}$ defined as $\bar{\omega} \equiv i^*\omega$ is a Cartan connection on $\bar{P}$, and ω is then said to be reducible to $\bar{\omega}$ on $i(\bar{P})$.

Moreover, if we denote

- $\hat{X} \in \Gamma(TP)$ *(resp. $\hat{\bar{X}} \in \Gamma(T\bar{P})$) the horizontal lift on P (resp. $\bar{P}$) of the vector field $X \in \Gamma(T\mathscr{M})$*
- $\varphi_p : T_{\pi(p)}\mathscr{M} \to \mathfrak{p}$ *(resp. $\bar{\varphi}_{\bar{p}} : T_{\bar{\pi}(\bar{p})}\mathscr{M} \to \mathfrak{p}$) the canonical isomorphism between the tangent space at $\mathscr{M}$ and the vector space of transvections $\mathfrak{p}$*
- $\Psi : \Gamma(T\mathscr{M}) \to \mathscr{T}(V,\rho)$ *(resp. $\bar{\Psi} : \Gamma(T\mathscr{M}) \to \bar{\mathscr{T}}(V,\rho)$) the canonical isomorphism between vector fields on $\mathscr{M}$ and tensors of type $\mathscr{T}(V,\rho)$ on P (resp. $\bar{\mathscr{T}}(V,\rho)$ on $\bar{P}$).*

then the following relations hold:

1. $i_*\hat{\bar{X}}_{\bar{p}} = \hat{X}_{i(\bar{p})}$
2. $\bar{\varphi}_{\bar{p}}\left(X_{\bar{\pi}(\bar{p})}\right) = \varphi_{i(\bar{p})}\left(X_{\pi\circ i(\bar{p})}\right)$
3. $\bar{\Psi}(X) = \Psi(X) \circ i$
4. $\hat{\bar{X}}\left[\bar{\Psi}(Y)\right] = \hat{X}\left[\Psi(Y)\right] \circ i$

$\forall\, X, Y \in \Gamma(T\mathscr{M})$.

In the reductive case i.e. when $\mathfrak{g}$ (resp. $\bar{\mathfrak{g}}$) admits the $\mathrm{Ad}(H)$-module (resp. $\mathrm{Ad}(\bar{H})$-module) decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ (resp. $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \mathfrak{p}$), one can define $\nabla : \Gamma(T\mathscr{M}) \to \Gamma(T\mathscr{M})$ (resp. $\bar{\nabla} : \Gamma(T\mathscr{M}) \to \Gamma(T\mathscr{M})$) the Cartan Koszul connection induced by (P,ω) (resp. $(\bar{P},\bar{\omega})$).

Then, one has:

5. $\bar{\nabla}_X Y = \nabla_X Y$, $\forall\, X, Y \in \Gamma(T\mathscr{M})$.

Proof: In order for $\bar{\omega}$ to be a Cartan connection, it must satisfy the axioms of Definition 5.2.1:

(a) Since the map i is an imbedding $i_* : T_{\bar{p}}\bar{P} \to T_{i(\bar{p})}P$ is injective. Furthermore, since $\bar{P}$ is an integral submanifold for $\mathcal{D}$, the vector spaces $T_{\bar{p}}\bar{P}$ and $\mathcal{D}_{i(\bar{p})}$ have same dimension so that $i_* : T_{\bar{p}}\bar{P} \to \mathcal{D}_{i(\bar{p})}$ is an isomorphism. Since, by definition of $\mathcal{D}$, $\omega_{i(\bar{p})} : \mathcal{D}_{i(\bar{p})} \to \bar{\mathfrak{g}}$ is an isomorphism, then, by composition, $\bar{\omega} : T_{\bar{p}}\bar{P} \to \bar{\mathfrak{g}}$ is an isomorphism.

(b) Denoting $R_h : P \to P$ (resp. $\bar{R}_{\bar{h}} : \bar{P} \to \bar{P}$) with $h \in H$ (resp. $\bar{h} \in \bar{H}$) the group action of H on P (resp. $\bar{H}$ on $\bar{P}$), the equivariance of ω with respect to $\bar{H}$ reads $R^*_{\varepsilon(\bar{h})}\omega = \mathrm{Ad}\left(\varepsilon\left(\bar{h}\right)^{-1}\right)\omega$ with $\bar{h} \in \bar{H}$. Applying i^* on both sides and acting on a vector field $\bar{X} \in \Gamma\left(T\bar{P}\right)$, one obtains:

$$\begin{aligned}
\left(i^* R^*_{\varepsilon(\bar{h})}\omega\right)\left(\bar{X}\right) &= \mathrm{Ad}\left(\varepsilon\left(\bar{h}\right)^{-1}\right) i^*\omega\left(\bar{X}\right)\\
\omega\left(R_{\varepsilon(\bar{h})*} i_* \bar{X}\right) &= \bar{\mathrm{Ad}}\left(\bar{h}^{-1}\right)\bar{\omega}\left(\bar{X}\right)\\
\omega\left(i_* R_{\bar{h}*}\bar{X}\right) &= \bar{\mathrm{Ad}}\left(\bar{h}^{-1}\right)\bar{\omega}\left(\bar{X}\right)\\
R^*_{\bar{h}}\bar{\omega}\left(\bar{X}\right) &= \bar{\mathrm{Ad}}\left(\bar{h}^{-1}\right)\bar{\omega}\left(\bar{X}\right)
\end{aligned}$$

where $\bar{\mathrm{Ad}} \equiv \mathrm{Ad} \circ \varepsilon$ is the adjoint action of $\bar{H}$ on $\bar{\mathfrak{h}}$. In the second step, the fact that i is a bundle homomorphism (*i.e.* $i\left(\bar{R}_{\bar{h}}\bar{p}\right) = R_{\epsilon(\bar{h})} i\left(\bar{p}\right)$, with $\bar{p} \in \bar{P}$) has been used.

(c) Let $\bar{X}^\sharp$ and $X^\sharp$ be the fundamental vector fields associated to the element $\bar{X} \in \bar{\mathfrak{h}}$ respectively on $\bar{P}$ and P. Let us first show that $X^\sharp_{i(\bar{p})} = i_* \bar{X}^\sharp_{\bar{p}}$:

$$\begin{aligned}
i_* \bar{X}^\sharp_{\bar{p}}(f) &= \bar{X}^\sharp_{\bar{p}}(f \circ i)\\
&= \left.\frac{d}{dt}\right|_{t=0} f \circ i\left(\bar{R}\left(\bar{p}, \exp\left(t\bar{X}\right)\right)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} f\left(R\left(i\left(\bar{p}\right), \varepsilon\left(\exp\left(t\bar{X}\right)\right)\right)\right)\\
&= X^\sharp_{i(\bar{p})}(f)
\end{aligned}$$

with $f \in C^\infty(P)$. Therefore $\bar{\omega}\left(\bar{X}^\sharp\right) = i^*\omega\left(\bar{X}^\sharp\right) = \omega\left(i_*\bar{X}^\sharp\right) = \omega\left(X^\sharp\right) = \bar{X}$.

We now establish relations *1-5*:

1. Since $\omega^{\mathfrak{h}}\left(i_*\hat{\bar{X}}\right) = i^*\omega^{\mathfrak{h}}\left(\hat{\bar{X}}\right) = \bar{\omega}^{\mathfrak{h}}\left(\hat{\bar{X}}\right) = 0$ and $\pi_* \circ i_*\hat{\bar{X}} = \bar{\pi}_*\hat{\bar{X}} = X$, then $i_*\hat{\bar{X}}$ coincides with $\hat{X}$ on $i\left(\bar{P}\right)$, *i.e.* $\hat{\bar{X}}$ and $\hat{X}$ are i-related (in the sense of Definition A.2.2).
2. We have $\bar{\varphi}_{\bar{p}}\left(X_{\bar{\pi}(\bar{p})}\right) = \bar{\omega}^{\mathfrak{p}}_{\bar{p}}\left(\hat{\bar{X}}_{\bar{p}}\right) = i^*\omega^{\mathfrak{p}}_{i(\bar{p})}\left(\hat{\bar{X}}_{\bar{p}}\right) = \omega^{\mathfrak{p}}_{i(\bar{p})}\left(\hat{X}_{i(\bar{p})}\right) = \varphi_{i(\bar{p})}\left(X_{\pi\circ i(\bar{p})}\right)$.
3. We only give the proof for vector fields, as it generalises easily to other tensors. Let $Y \in \Gamma\left(T\mathscr{M}\right)$ be a vector field. Then, from relation *2*, one obtains

$\bar{\Psi}(Y)(\bar{p}) \equiv \bar{\varphi}_{\bar{p}}\left(Y_{\bar{\pi}(\bar{p})}\right) = \varphi_{i(\bar{p})}\left(Y_{\pi\circ i(\bar{p})}\right) = \Psi(Y) \circ i(\bar{p})$.

4. From relation *1*, *3* and Proposition A.2.3, one obtains straightforwardly $\hat{\bar{X}}\left[\bar{\Psi}(Y)\right] = \hat{\bar{X}}\left[\Psi(Y)\circ i\right] = \hat{X}\left[\Psi(Y)\right]\circ i$.
5. Starting from the definition of the Cartan derivative and using relations *3* and *4* leads to $\bar{\Psi}\left(\bar{\nabla}_X Y\right) = \hat{\bar{X}}\left[\bar{\Psi}(Y)\right] = \hat{X}\left[\Psi(Y)\right]\circ i = \Psi\left(\nabla_X Y\right)\circ i = \bar{\Psi}\left(\nabla_X Y\right)$. Recalling that $\bar{\Psi}$ is an isomorphism concludes the proof.

□

Applying Proposition 6.2.1 to the case at hand ensures that the 1-form $\bar{\omega} \equiv i^*\omega \in \Omega^1\left(\bar{P}\right)\otimes\bar{\mathfrak{g}}$ is a Cartan-Galilei connection for the bundle $\bar{H} \hookrightarrow \bar{P} \to \mathscr{M}$, so that $\left(\bar{P}, \bar{\omega}\right)$ is a well-defined Cartan-Galilei geometry embedded into the Cartan-Bargmann geometry (P, ω).

We now investigate the consequences of the involutivity of the distribution $\mathcal{D}$. According to Proposition 5.3.22, the distribution $\mathcal{D}$ is involutive on P if and only if, for all $p \in P$, the curvature 2-form satisfies $\Omega_p\left(X_p, Y_p\right) \in \mathfrak{gal}$, $\forall X_p, Y_p \in T_pP$. In order to get things more concrete, we perform the following expansion of the torsionfree Cartan-Bargmann connection and curvature on the canonical basis for $\mathfrak{bar}$ and $\mathfrak{gal}_o \oplus \mathbb{R}$, respectively:

$$\begin{aligned} \omega &= \psi H + \theta^i P_i + \omega^{ij} J_{ij} + \omega^i K_i + AM \\ \Omega &= R^{ij} J_{ij} + R^i K_i + R^M M \end{aligned}$$

where $\psi, \theta^i, \omega^{ij}, \omega^i$ and A are (sets of) 1-forms on P and R^{ij}, R^i, R^M (sets of) 2-forms on P. The involutivity condition can then be restated by imposing that the curvature 2-form does not take value in the mass part M of the Bargmann algebra, *i.e.*

$$R^M = 0. \tag{6.2.19}$$

We are now looking to make contact between the involutivity condition (6.2.19) and the Duval-Künzle condition (*cf.* Section 4 of [86] for an analogous computation). In order to do this, we compute the "mass"-part of the (second) Bianchi identity (*cf.* eq.(5.3.16)) which reads:

$$dR^M - \frac{1}{2}R^i \wedge \theta_i = 0$$

so that we have the following implication:

$$\mathcal{D} \text{ is involutive } \Rightarrow R^i \wedge \theta_i = 0.$$

The proposition on the right-hand side descends also to the "boost"-part $\bar{R}^i \in \Omega^2\left(\bar{P}\right)$ (as can be seen straightforwardly by application of the pullback i^*) so that the involutivity of $\mathcal{D}$ implies $\bar{R}^i \wedge \bar{\theta}_i = 0$. Choosing a gauge $\left(\mathcal{U}, \bar{\theta}\right)$ with $\mathcal{U} \subset \mathscr{M}$ an open subset of $\mathscr{M}$ and $\bar{\theta} \in \Omega^1\left(\mathscr{M}\right) \otimes \mathfrak{bar}$ the gauge-connection associated to the section $\bar{\sigma} : \mathcal{U} \to \bar{P}$, one can make use of eq.(6.1.15) in order to reexpress the gauge expression of the boost curvature $\bar{R}^i \in \Omega^1\left(\mathscr{M}\right)$ (conserving the same symbol for bundle objects and their gauge avatars for notational simplicity) as $\bar{R}^i = \bar{R}^i{}_0 = R^\lambda{}_\rho N^\rho \theta^i_\lambda$ where $\bar{R}^i{}_0 \in \Omega^2\left(\mathscr{M}\right)$ stands for the spatio-temporal part of the curvature $R^\lambda{}_\rho \in \Omega^2\left(\mathscr{M}\right)$ for the Cartan-Galilei Koszul connection $\bar{\nabla}$ as expressed in the Galilei basis defined by the gauge $\left(\mathcal{U}, \bar{\theta}\right)$ (*cf.* Section 6.1). Consequently, the gauge expression of the involutivity condition:

$$\bar{R}^i{}_0 \wedge \bar{\theta}^i = 0 \tag{6.2.20}$$

is just the Duval-Künzle condition in the formulation of Proposition 3.2.29.

We sum up this Section by the following Proposition:

Proposition 6.2.2 ([53, 86])**.** *The base space of a Cartan-Galilei geometry embedded in a torsionfree Cartan-Bargmann geometry is a Newtonian manifold.*

where the invoked notion of embedding is to be understand in the sense of Proposition 6.2.1.

6.3 Bargmann-Eisenhart waves as reductive Cartan-Bargmann geometries

The two previous Sections advocated the relevance of the Cartanian approach regarding the study of intrinsically nonrelativistic structures and manifolds by emphasising their group-theoretical origins. In the present Section, we pursue this leitmotif with the aim to shed some light on certain features regarding the ambient formalism, as developed in Chapter 4. This reformulation of known results will make crucial use of the Bargmann group, whose pertinence as regards the definition of Newtonian manifolds has been highlighted in the last Section. This is only natural in view of the heuristic discussion of Section 3.2.2 where Newtonian manifolds were argued to provide the best hint for the existence of an additional light-like direction.

We thus again consider a Cartan-Bargmann geometry, but with a key difference as compared with the construction of Section 6.2. Indeed, the mass element M of the Bargmann algebra was there considered as part as the "homogeneous"-part $\mathfrak{h}$ of the Lie algebra $\mathfrak{bar}$, as was natural since the stabiliser of a point is $\mathrm{Gal}_0 \times \mathbb{R}$. Compared to a Galilei geometry (*cf.* Section 6.1), this choice presented the advantage to preserve the dimension of the base manifold, as it should to describe a nonrelativistic manifold. However, a drawback of this approach lied in the fact that the Cartan-Bargmann geometry thus defined was not reductive.

In contradistinction with the previous case, the model Klein geometry of the present construction is chosen to be $(\mathfrak{bar}, \mathfrak{gal}_o)$, with $\mathfrak{bar}$ the Bargmann algebra and $\mathfrak{gal}_o$ the homogeneous Galilei algebra. It can be checked that this Klein geometry is reductive and as such admits the following $\mathrm{Ad}\,(\mathrm{Gal}_0)$-module decomposition: $\mathfrak{bar} = \mathfrak{gal}_o \oplus \mathfrak{p}$ where $\mathfrak{p} = \mathrm{Span}\,\{H, P_i, M\}$. The generator M has thus migrated from the homogeneous part $\mathfrak{h}$ to the translation part $\mathfrak{p}$ which results in $[P_i, K_j] = i\delta_{ij} M \subset \mathfrak{p}$, so that $\mathfrak{p}$ is an $\mathrm{ad}\,(\mathfrak{h})$-module.

Now, let (P, ω) be a Cartan-Bargmann geometry modeled on the infinitesimal Klein pair $(\mathfrak{bar}, \mathfrak{gal}_o)$, so that P is a Gal_0-bundle with base manifold $\mathscr{M}$ and associated projection $\pi : P \to \mathscr{M}$:

$$
\begin{tikzcd}
\mathrm{Gal}_0 \arrow[d, hook] \\
P \arrow[d, "\pi"] \\
\mathscr{M}
\end{tikzcd}
$$

where the manifold P is endowed with a Cartan connection $\omega \in \Omega^1(\mathscr{M}) \otimes \mathfrak{bar}$ taking values in the Bargmann algebra.

Similarly to the analysis of Section 6.1, we will characterise the base manifold $\mathscr{M}$ by investigating $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\,(\mathrm{Gal}_0)$-invariant elements of $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$ and then build up their corresponding structures on $\mathscr{M}$. Using the representation theory of the Bargmann group (*cf.* Section B.2.3), one can single out the following elements

- $M \in \mathfrak{p}$
- $H^* \in \mathfrak{p}^*$
- $(\cdot,\cdot)_{\mathfrak{p}^*} \equiv 2M^* \vee H^* + \delta_{ij} P^{i*} \vee P^{j*} \in \vee^2 \mathfrak{p}^*$
- $(\cdot,\cdot)_{\mathfrak{p}} \equiv 2M \vee H + \delta^{ij} P_i \vee P_j \in \vee^2 \mathfrak{p}$

as the only $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\,(\mathrm{Gal}_0)$-invariant elements of $\bigotimes^p \mathfrak{p} \otimes \bigotimes^q \mathfrak{p}^*$, with $p + q \leq 2$. Comparing these invariant elements to the ones derived in the Galilean case, one notices that the presence of the central mass element M allowed the construction of a nondegenerate covariant bilinear form $(\cdot,\cdot)_{\mathfrak{p}^*} \in \vee^2 \mathfrak{p}^*$, along with its contravariant inverse. This nondegenerate metric structure ensures that we are dealing with a Riemannian geometry, in contradistinction with the nonrelativistic metric structure emerging from the non-extended Galilean group. For future use, we introduce also the two non-invariant bilinear forms on $\mathfrak{p}$:

- $(\cdot,\cdot)^{\perp}_{\mathfrak{p}^*} \equiv \delta_{ij} P^{i*} \vee P^{j*} \in \vee^2 \mathfrak{p}^*$
- $(\cdot,\cdot)^{\perp}_{\mathfrak{p}} \equiv \delta^{ij} P_i \vee P_j \in \vee^2 \mathfrak{p}$.

Now, mirroring the procedure outlined in Chapter 5 and employed in Section 6.1, we first make use of Proposition A.6.10 in order to convert these $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Gal}_0)$-invariant elements into tensors on P and then lower these tensors downto canonical fields on $\mathscr{M}$ using the isomorphism $\Psi^{-1} : \mathscr{T}(V, \rho) \to \Gamma(E)$ (*cf.* Definition 5.2.13 and Table 5.3). These are summarised in the following Table:

Symbol	Type	Root	Type	Definition
ξ	$\Gamma(T\mathscr{M})$	M	$\mathfrak{p}$	$\xi \equiv \Psi^{-1}(M)$
ψ	$\Omega^1(\mathscr{M})$	H^*	$\mathfrak{p}^*$	$\psi \equiv \bar{\Psi}^{-1}(H^*)$
g	$\Gamma\left(\vee^2 T\mathscr{M}\right)$	$(\cdot,\cdot)_{\mathfrak{p}^*}$	$\vee^2\,\mathfrak{p}^*$	$g \equiv \bar{\bar{\Psi}}^{-1}(\cdot,\cdot)_{\mathfrak{p}^*}$

Table 6.1: Canonical fields induced on the base space of a reductive Cartan-Bargmann geometry

Several interesting properties regarding these canonical fields can be deduced from their Lie algebra ascendants. First of all, the vector field $\xi \in \Gamma(T\mathscr{M})$ can be seen to be null with respect to the metric g, as a consequence of the "light-like" character of M with respect to $(\cdot,\cdot)_{\mathfrak{p}^*}$:

$$g(\xi, \xi) = (\Psi(\xi), \Psi(\xi))_{\mathfrak{p}^*} = (M, M)_{\mathfrak{p}^*} = 0. \tag{6.3.21}$$

Secondly, inspection of the form of the metric $(\cdot,\cdot)_{\mathfrak{p}^*}$ reveals that the elements M and H^* enjoy a dual relation, as displayed by the relation $(M,\cdot)_{\mathfrak{p}^*} = H^*$. This dual relation passes on ξ and ψ as:

$$g(\xi, X) = (\Psi(\xi), \Psi(X))_{\mathfrak{p}^*} = (M, \Psi(X))_{\mathfrak{p}^*} = H^*(\Psi(X)) = \psi(X), \ \forall X \in \Gamma(T\mathscr{M})$$

so that $\psi = g(\xi)$.

We now make use of the fact that the Klein geometry modeling our Cartan geometry is reductive in order to induce a Cartan-Koszul connection ∇ on the base space (*cf.* Propositions 5.3.13 and 5.3.14). According to Proposition 5.3.15, the Cartan-Koszul connection ∇ has the nice property to preserve the fields on $\mathscr{M}$ which originate from constant $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$-invariant elements of $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$. In particular, the nondegenerate metric g is thus preserved by ∇. Making the further assumption that our Cartan-Bargmann geometry is torsionfree, one concludes from Theorem 3.1.5 that ∇ is the Levi-Civita connection associated to the (pseudo)-Riemannian metric g endowing $\mathscr{M}$. The same line of reasoning ensures that both ξ and ψ are parallelised by ∇. The base space $\mathscr{M}$ thus admits a null and parallel vector field, so that the following Proposition holds:

Proposition 6.3.1 (*cf.* [24])**.** *The base manifold of a Cartan-Bargmann geometry modeled on the infinitesimal Klein pair* $(\mathfrak{bar}, \mathfrak{gal}_0)$ *is a Bargmann-Eisenhart wave.*

In the light of the previous Proposition, the vector field ξ takes the interpretation of the wave vector field of $\mathscr{M}$ (similarly, ψ is understood as the wave covector field). The study of the properties regarding $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Gal}_0)$-invariant elements of $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$ is seen to be sufficient in order to characterise the base manifold as a Bargmann-Eisenhart wave.

Beyond the identification between $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Gal}_0)$-invariant elements of $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$ and canonical fields on $\mathscr{M}$, the Cartanian approach has been seen to provide non-invariant elements of $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$ with the interpretation of gauge-dependent fields living on the base space, whose transformation law under a change of section is inherited from the $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Gal}_0)$ transformation law of the associated tensors on $\bigotimes \mathfrak{p} \otimes \bigotimes \mathfrak{p}^*$. Such an identification necessitates the introduction of a section $\sigma : \mathscr{U} \subset \mathscr{M} \to P$, with $\mathscr{U}$ an open set of $\mathscr{M}$. We let again $\overset{\sigma}{\Upsilon}{}^{-1} : C^\infty(\mathscr{U}) \otimes \mathfrak{p} \to \Gamma(T\mathscr{U})$ designate the (section dependent) map between functions on $\mathscr{U}$ taking values in $\mathfrak{p}$ and vector fields on $\mathscr{U}$ defined as $\overset{\sigma}{\Upsilon}{}^{-1}(\phi)(x) = \varphi^{-1}_{\sigma(x)}(\phi(x))$, with $\phi \in C^\infty(\mathscr{U}) \otimes \mathfrak{p}$.

In addition to the canonically defined fields of Table 6.1, the map $\overset{\sigma}{\Upsilon}{}^{-1}$ allows the definition of the following (gauge-dependent) fields:

Symbol	Type	Root	Type	Definition
N	$\Gamma(T\mathscr{M})$	H	$\mathfrak{p}$	$N \equiv \overset{\sigma}{\Upsilon}{}^{-1}(H)$
$\overset{N}{e}_i$	$\Gamma(T\mathscr{M})$	P_i	$\mathfrak{p}$	$\overset{N}{e}_i \equiv \overset{\sigma}{\Upsilon}{}^{-1}(P_i)$
$\overset{N}{A}$	$\Omega^1(\mathscr{M})$	M^*	$\mathfrak{p}^*$	$\overset{N}{A} \equiv \overset{\sigma}{\Upsilon}{}^{-1}(M^*)$
$\overset{N}{\theta}{}^i$	$\Omega^1(\mathscr{M})$	P^{i*}	$\mathfrak{p}^*$	$\overset{N}{\theta}{}^i \equiv \overset{\sigma}{\Upsilon}{}^{-1}\left(P^{i*}\right)$

Table 6.2: Gauge-dependent fields induced on the base space of a reductive Cartan-Bargmann geometry

Symbol	Type	Root	Type	Definition
$\overset{N}{\gamma}$	$\Gamma\left(\vee^2 T^*\mathscr{M}\right)$	$(\cdot,\cdot)^{\perp}_{\mathfrak{p}^*}$	$\vee^2\,\mathfrak{p}^*$	$\overset{N}{\gamma}\equiv\overset{\sigma}{\Upsilon}^{-1}(\cdot,\cdot)^{\perp}_{\mathfrak{p}^*}$
$\overset{N}{h}$	$\Gamma\left(\vee^2 T\mathscr{M}\right)$	$(\cdot,\cdot)^{\perp}_{\mathfrak{p}}$	$\vee^2\,\mathfrak{p}$	$\overset{N}{h}\equiv\overset{\sigma}{\Upsilon}^{-1}(\cdot,\cdot)^{\perp}_{\mathfrak{p}}$

Table 6.3: Gauge-dependent metrics

Again, using the structure of the Lie algebra $\mathfrak{bar}$ (more specifically the module $\mathfrak{p}$) allows to further characterise these gauge-dependent fields and thus to make contact with the related notions introduced in Section 4.3. The following Proposition fits into this framework:

Proposition 6.3.2. *The base space of a reductive Cartan-Bargmann geometry is endowed with a non-canonical Bargmann basis imported from the canonical basis for the module* $\mathfrak{p}$*.*

Proof: We start by showing that the vector field N is a relativistic field of light-like observers, *i.e.* that it satisfies the relations $g(N,N)=0$, $\psi(N)=1$. Both these relations can be deduced in a straightforward way as follows:

- $g(N,N)=\left(\overset{\sigma}{\Upsilon}(N),\overset{\sigma}{\Upsilon}(N)\right)_{\mathfrak{p}^*}=(H,H)_{\mathfrak{p}^*}=0$
- $\psi(N)=H^*\left(\overset{\sigma}{\Upsilon}(N)\right)=H^*(H)=1.$

Note that the vector field N and the 1-form $\overset{N}{A}$ are dual with respect to the metric g:

$$g(N,X)=\left(\overset{\sigma}{\Upsilon}(H),\overset{\sigma}{\Upsilon}(X)\right)_{\mathfrak{p}^*}=\left(H,\overset{\sigma}{\Upsilon}(X)\right)_{\mathfrak{p}^*}=\overset{N}{A}(X)\,,\ \forall\, X\in\Gamma(T\mathscr{M})$$

so that $\overset{N}{A}=g(N)$. Secondly, it can be shown that the set of vector fields $\overset{N}{e}_i$, with $i\in\{1,\ldots,d\}$ form a basis of the subbundle Ker $\psi\cap$ Ker $\overset{N}{A}$ which is orthonormal with respect to g. That the vector fields $\overset{N}{e}_i$ belong to Ker $\psi\cap$ Ker $\overset{N}{A}$ follows from:

- $\psi\left(\overset{N}{e}_i\right)=H^*\left(\overset{\sigma}{\Upsilon}\left(\overset{N}{e}_i\right)\right)=H^*(P_i)=0$
- $\overset{N}{A}\left(\overset{N}{e}_i\right)=M^*\left(\overset{\sigma}{\Upsilon}\left(\overset{N}{e}_i\right)\right)=M^*(P_i)=0$

while the fact they form a basis is ensured by the fact that the elements P_i form a basis of Ker $H^*\cap$ Ker M^*. Moreover, this basis is orthonormal with respect to g, as the following computation shows:

$$g\left(\overset{N}{e}_i,\overset{N}{e}_j\right)=\left(\overset{\sigma}{\Upsilon}\left(\overset{N}{e}_i\right),\overset{\sigma}{\Upsilon}\left(\overset{N}{e}_j\right)\right)_{\mathfrak{p}^*}=\left(P_i,P_j\right)_{\mathfrak{p}^*}=\delta_{ij}.$$

This concludes the proof of the fact that $B\equiv\left\{\xi,N,\overset{N}{e}_i\right\}$ is a Bargmann basis of the tangent bundle T_M. Accordingly, $B^*\equiv\left\{\overset{N}{A},\psi,\overset{N}{\theta}{}^i\right\}$ is a (dual) Bargmann basis of $T^*\mathscr{M}$. $\square$

As was noted in Proposition 4.3.10, at each point $x\in\mathscr{M}$ the set of endomorphisms of $T_x\mathscr{M}$ mapping each Bargmann basis into another one forms a group isomorphic to the homogeneous Galilei group Gal_0, in the $d+2$-dimensional faithful representation inherited from that of the Bargmann group. That this is the case is clear in the present formalism, since a change of basis takes now the interpretation of a gauge-transformation *i.e.* a change of section $\sigma\overset{h}{\Rightarrow}\sigma'$, where $h:\mathscr{U}\cap\mathscr{V}\to\mathrm{Gal}_0$ is a function taking values in the homogeneous Galilei group. Moreover, the right-action R_h being regular on the fibers of P ensures that the homogeneous Galilei group acts regularly on the space of Bargmann basis (again in the representation (6.2.16)).

We now compute the explicit transformation of N under a change of section $h:\mathscr{U}\cap\mathscr{V}\to\mathrm{Gal}_0$, with h given by

$$h=\begin{pmatrix}1&0&0\\-\frac{1}{2}\mathbf{b}^T\mathbf{b}&1&\mathbf{b}\\-\mathbf{b}^T\mathbf{R}&0&\mathbf{R}\end{pmatrix}\tag{6.3.22}$$

with $\mathbf{R}:\mathscr{M}\to O(d)$ and $\mathbf{b}^i:\mathscr{M}\to\mathbb{R}^d$. We will make use of the transformation law $\overset{\sigma'}{\Upsilon}{}^{-1}(\phi)=\overset{\sigma}{\Upsilon}{}^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,\phi\right)$, with $\phi\in C^\infty(\mathscr{U})\otimes\mathfrak{p}$. Focusing on the relativistic field of light-like observers N, we compute:

$$N'\equiv\overset{\sigma'}{\Upsilon}{}^{-1}(H)=\overset{\sigma}{\Upsilon}{}^{-1}\left(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,H\right)=\overset{\sigma}{\Upsilon}{}^{-1}\left(H+\mathbf{b}^iP_i-\frac{1}{2}\mathbf{b}^T\mathbf{b}\,M\right)=N+\mathbf{b}^i\overset{N}{e}_i-\frac{1}{2}\mathbf{b}^T\mathbf{b}\,\xi.$$

Under a change of gauge parameterised by the function h given in eq.(6.3.22), the relativistic field of light-like observers N is thus mapped to the relativistic field of light-like observers N' via a relativistic Milne boost parameterised by the 1-form $\chi\equiv\mathbf{b}^ie_i$ (*cf.* Proposition 4.3.14). For the sake of exhaustivity, the following Table compiles the transformation laws enjoyed by the fields of Table 6.2:

Field	Root	$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)$-transformation	Gauge-transformation
N	H	$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,H = H + \mathbf{b}^i P_i - \frac{1}{2}\mathbf{b}^T\mathbf{b}\,M$	$N' = N + \mathbf{b}^i \overset{N}{e}_i - \frac{1}{2}\mathbf{b}^T\mathbf{b}\,\xi$
$\overset{N}{e}_i$	P_i	$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\,P_i = \mathbf{R}^j{}_i P_j - \mathbf{b}_j^T \mathbf{R}^j{}_i M$	$\overset{N'}{e}_i = \mathbf{R}^j{}_i \overset{N}{e}_j - \mathbf{b}_j^T\mathbf{R}^j{}_i\xi$
$\overset{N}{A}$	M^*	$\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)\,M^* = M^* - \frac{1}{2}\mathbf{b}^T\mathbf{b}\,H^* + \mathbf{b}_i^T P^{i*}$	$\overset{N'}{A} = \overset{N}{A} - \frac{1}{2}\mathbf{b}^T\mathbf{b}\,\psi + \mathbf{b}_i^T \overset{N}{\theta}{}^i$
$\overset{N}{\theta}{}^i$	P^{i*}	$\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)\,P^{i*} = \mathbf{R}^{T\,i}{}_j P^{j*} - \mathbf{R}^{T\,i}{}_j \mathbf{b}^j H^*$	$\overset{N'}{\theta}{}^i = \mathbf{R}^{T\,i}{}_j \overset{N}{\theta}{}^j - \mathbf{R}^{T\,i}{}_j\mathbf{b}^j\psi$

Table 6.4: Gauge transformations

Lemma 6.3.3. *Let us denote $\hat{\xi} \in \Gamma(TP)$ the horizontal lift of the wave vector field. The vector field $\left[\hat{\xi}, \hat{X}\right] \in \Gamma(TP)$ is horizontal for every horizontal vector field $\hat{X} \in HP$.*

Proof: Since the wave vector field ξ is parallelised by ∇, the Koszul curvature associated to ∇ satisfies $R(Y,Z;\xi) = 0$, $\forall Y, Z \in \Gamma(T\mathscr{M})$. Using the symmetry relation (A.9.18), one concludes that $\forall X, Y, Z \in \Gamma(T\mathscr{M})$, $0 = R(Y,Z;\xi,X) = R(\xi,X;Y,Z) = g(R(\xi,X;Y), Z)$. Since the Cartan metric $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ is non-degenerate, this implies $R(\xi,X;Y) = 0$, $\forall Y, Z \in \Gamma(T\mathscr{M})$. Using Proposition 5.3.19 leads to $\left[\hat{\xi}, \hat{X}\right]^V = 0$, $\forall X \in \Gamma(T\mathscr{M})$, where $\hat{X} \in \Gamma(TP)$ is the horizontal lift of X and the exponent V designates the vertical part of the vector field.

Given a horizontal vector field $\hat{Y} \in HP$, it is always possible at each point $p \in P$ to find a base vector field $Y \in \Gamma(T\mathscr{M})$ such that $\pi_*\hat{Y}_p = Y_{\pi(p)}$ *i.e.* any horizontal vector field can be locally interpreted as the horizontal lift of a base vector field. The local character of the Lie bracket then ensures that $\left[\hat{\xi}, \hat{Y}\right]^V = 0$ holds for any horizontal vector field $\hat{Y} \in HP$. □

Lemma 6.3.4. *The horizontal lift of the wave vector field is an infinitesimal automorphism of ω, i.e. $\mathcal{L}_{\hat{\xi}}\omega = 0$.*

Proof: Since $\hat{\xi}$ is by construction an ω-constant vector field (*cf.* Definition 5.3.20), Proposition 5.3.21 ensures that $\forall Z \in \Gamma(TP)$, $\mathcal{L}_{\hat{\xi}}\omega(Z) = [\omega(Z), M] + \Omega\left(\hat{\xi}, Z\right) = \Omega^{\mathfrak{h}}\left(\hat{\xi}, Z^H\right)$ where we used that the geometry is torsion-free ($\Omega^{\mathfrak{p}} = 0$) as well as the horizontality of the curvature (*cf.* Proposition 5.2.4). Now, applying relation (5.3.14) leads to $\mathcal{L}_{\hat{\xi}}\omega\left(Z\right)=d\omega^{\mathfrak{h}}\left(\hat{\xi},Z^H\right)=\hat{\xi}\left[\omega^{\mathfrak{h}}\left(Z^H\right)\right]-Z^H\left[\omega^{\mathfrak{h}}\left(\hat{\xi}\right)\right]-\omega^{\mathfrak{h}}\left(\left[\hat{\xi},Z^H\right]\right)$. On the right-hand side, the first term vanishes due to the horizontality of Z^H while the second term is null since $\hat{\xi}$ is ω-constant. Finally, Lemma 6.3.3 ensures that the third term vanishes, so that $\mathcal{L}_{\hat{\xi}}\omega\left(Z\right)=0$, $\forall Z\in\Gamma\left(TP\right)$. □

Proposition 6.3.5. *Let (P,ω) be a reductive Cartan-Bargmann geometry with $(\mathcal{U},\theta)$ a gauge associated to the section $\sigma:\mathscr{M}\to P$. Furthermore, let $x\in\mathfrak{p}$ a vector on $\mathfrak{p}$. Then the vector field $X\equiv\overset{\sigma}{\Upsilon}{}^{-1}\left(x\right)\in\Gamma\left(T\mathscr{M}\right)$ satisfies $\mathcal{L}_\xi X=0$.*

Proof: Let $Y\in\Gamma\left(T\mathscr{M}\right)$ be a vector field on $\mathscr{M}$. Applying Lemma 6.3.4 on the vector field $\hat{Y}\equiv\sigma_*Y\in\Gamma\left(TP\right)$ leads to $\left(\mathcal{L}_{\hat{\xi}}\omega\right)\left(\hat{Y}\right)=d\omega\left(\hat{\xi},\hat{Y}\right)=d\theta\left(\xi,Y\right)=\left(\mathcal{L}_\xi\theta\right)\left(Y\right)=0$, so that $\mathcal{L}_\xi\theta=0$. Now, comparing $\mathcal{L}_\xi\left(\theta\left(X\right)\right)=\left(\mathcal{L}_\xi\theta\right)\left(X\right)+\theta\left(\mathcal{L}_\xi X\right)=\theta\left(\mathcal{L}_\xi X\right)$ with $\mathcal{L}_\xi\left(\theta\left(X\right)\right)=\mathcal{L}_\xi x=0$ leads to $\theta\left(\mathcal{L}_\xi X\right)=0$, hence $\mathcal{L}_\xi X=0$. □

The previous Proposition generalises straightforwardly to any field of $\mathscr{M}$ originating from an element of $\bigotimes\mathfrak{p}\otimes\bigotimes\mathfrak{p}^*$. In the language of Chapter 5, such a field is thus called ξ-invariant. In particular, the vector fields composing the non-canonical Bargmann basis $B=\left\{\xi,N,\overset{N}{e}_i\right\}$ defined in Proposition 6.3.2 are ξ-invariant and thus admit a well-defined projection on the Platonic screen. The projection of the wave vector field ξ vanishes so that $\pi_*B=\left\{\bar{N},\overset{\bar{N}}{e}_i\right\}$. It can be checked that π_*B is a Galilean basis on the Platonic screen $\bar{\mathscr{M}}$. Accordingly, if we denote $B^*\equiv\left\{\overset{N}{A},\psi,\overset{N}{\theta}{}^i\right\}$, the equivariance of ψ and $\overset{N}{\theta}{}^i$ in addition to the relations $\psi\left(\xi\right)=\overset{N}{\theta}{}^i\left(\xi\right)=0$ ensures that the dual Bargmann basis projects onto the dual Galilean basis $\left\{\bar{\psi},\overset{\bar{N}}{\theta}{}^i\right\}$. The relation $\overset{N}{A}\left(\xi\right)=1$ prevents $\overset{N}{A}$ to project onto $\mathscr{M}$. However, this relation along with its equivariance ensures that it is a well-defined Ehresmann connection on the $(\mathbb{R},+)$-principal bundle $\mathbb{R}\hookrightarrow\mathscr{M}\to\bar{\mathscr{M}}$.

6.4 The ambient approach revisited

We conclude this Chapter by investigating how the ambient formalism takes shape in the Cartanian framework. This will be achieved by precising the link between the reductive and non-reductive versions of Cartan-Bargmann geometries. Again, we let $\mathrm{Gal}_0\hookrightarrow P\to\mathscr{M}$ be a torsionfree Cartan-Bargmann geometry modeled on the infinitesimal reductive Klein pair $(\mathfrak{bar},\mathfrak{gal}_0)$. The right-action of Gal_0 on P is denoted $R:P\times\mathrm{Gal}_0\to P$, so that the base space $\mathscr{M}$ is the quotient of P by the right-action R. Proposition 6.3.1 ensures that $\mathscr{M}$ is a Bargmann-Eisenhart wave while the origin of the wave vector field ξ can be traced back to the "mass" generator of the Gal_0-module $\mathfrak{p}$.

Similarly to the construction performed in Section 4.1, one can define the right action of $(\mathbb{R},+)$ on $\mathscr{M}$, denoted $\varphi_\xi : \mathscr{M} \times \mathbb{R} \to \mathscr{M}$, as the flow generated by the wave vector field [2]. Quotienting the base manifold $\mathscr{M}$ with the right-action φ_ξ yields a $(\mathbb{R},+)$-principal bundle whose base space is the Platonic screen $\bar{\mathscr{M}}$ of the Bargmann-Eisenhart wave $\mathscr{M}$.

$$\begin{tikzcd} & \mathrm{Gal}_0 \arrow[d, hook, "R"] & \\ & P \arrow[d, "\pi"] & \\ \bar{\mathscr{M}} & \mathscr{M} \arrow[l, "\bar{\pi}"'] & \mathbb{R} \arrow[l, hook', "\varphi_\xi"'] \end{tikzcd} \tag{6.4.23}$$

Let us denote $\hat{\xi} \in \Gamma(TP)$ the horizontal lift of the wave vector field with respect to the (reductive) Cartan-Bargmann connection $\omega \in \Omega^1(P) \otimes \mathfrak{bar}$. Similarly to its base counterpart, $\hat{\xi}$ defines a right-action of $(\mathbb{R},+)$ on P, denoted $\hat{\varphi}_\xi : P \times \mathbb{R} \to P$. The manifold P is then endowed with the two right-actions R and $\hat{\varphi}_\xi$. The equivariance of ξ ensures that its flow is a bundle isomorphism (*cf.* Proposition 5.3.24), so that $\hat{\varphi}_\xi$ and R commute (roughly speaking, $\hat{\varphi}_\xi$ is "horizontal" while R is "vertical"). These two right-actions on P can thus be combined to form the right action $\mathscr{R}$ on P for the direct product $\mathrm{Gal}_0 \times \mathbb{R}$ acting as $\mathscr{R} : P \times (\mathrm{Gal}_0 \times \mathbb{R}) \to P : (p,(h,\lambda)) \mapsto R_h(\hat{\varphi}_\xi(p,\lambda))$. The group operation of the direct product $\mathrm{Gal}_0 \times \mathbb{R}$ reads explicitly $(h_1,\lambda_1).(h_2,\lambda_2) = (h_1h_2, \lambda_1+\lambda_2)$, where $h_1, h_2 \in \mathrm{Gal}_0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. The fact that $\hat{\varphi}_\xi$ is a bundle automorphism ensures that $\mathscr{R}$ preserves the previous group operation, *i.e.* $\mathscr{R}_{(h_2,\lambda_2)}\mathscr{R}_{(h_1,\lambda_1)}p = \mathscr{R}_{(h_1h_2,\lambda_1+\lambda_2)}p$, $\forall\, p \in P$.

The crucial point of the reasoning lies in the reinterpretation of the horizontal vector field $\hat{\xi}$ in the principal bundle $\mathrm{Gal}_0 \overset{R}{\hookrightarrow} P \to \mathscr{M}$, as a fundamental vector field for the principal bundle $\mathrm{Gal}_0 \times \mathbb{R} \overset{\mathscr{R}}{\hookrightarrow} P \to \bar{\mathscr{M}}$, where the Platonic screen $\bar{\mathscr{M}}$ is reinterpreted as the quotient manifold of P by the right-action $\mathscr{R}$.

This reinterpretation can be reformulated more pictorially by the commuting of the following diagram:

$$\begin{tikzcd} \mathrm{Gal}_0 \arrow[rr, hook] \arrow[dr, hook, "R"'] & & \mathrm{Gal}_0 \times \mathbb{R} \arrow[dl, hook', "\mathscr{R}"] & \\ & P \arrow[dl] \arrow[dr, "\pi"] & & \\ \bar{\mathscr{M}} & & \mathscr{M} \arrow[ll, "\bar{\pi}"'] & \mathbb{R} \arrow[l, hook', "\varphi_\xi"'] \end{tikzcd}$$

2. As in Section 4.1, ξ is assumed to be complete and to define a free and proper right-action φ_ξ on $\mathscr{M}$.

The Lie algebra counterpart of this substitution clearly consists in the trading of the "mass" generator M from the "transvection part" $\mathfrak{p}$ to the "homogeneous" part $\mathfrak{h}$ of the Bargmann algebra $\mathfrak{bar}$. As noted earlier, the price to be paid is that the new Lie algebra is not reductive.

In order for the picture to be complete, it remains to be proved that ω is also a Cartan-Bargmann connection for the bundle $\mathrm{Gal}_0 \times \mathbb{R} \hookrightarrow P \to \bar{\mathscr{M}}$.

Proposition 6.4.1. *The 1-form* $\omega \in \Omega^1(P) \otimes \mathfrak{bar}$ *is a Cartan connection for the bundle* $\mathrm{Gal}_0 \times \mathbb{R} \hookrightarrow P \to \bar{\mathscr{M}}$.

Proof: Since ω is a connection for the bundle $\mathrm{Gal}_0 \hookrightarrow P \to \mathscr{M}$, by Definition 5.2.1, the following properties hold:

(a) for each point $p \in P$, the linear map $\omega_p : T_pP \to \mathfrak{bar}$ is an isomorphism

(b) $(R_h)^* \omega = \mathrm{Ad}\left(h^{-1}\right) \omega$, $\forall h \in \mathrm{Gal}_0$ where $\mathrm{Ad} : \mathrm{Gal}_0 \to \mathrm{End}\left(\mathfrak{bar}\right)$ stands for the adjoint action of Gal_0 on $\mathfrak{bar}$

(c) $\omega\left(X^\sharp\right) = X$, $\forall X \in \mathfrak{gal}_0$ where $X^\sharp \in VP$ designates the fundamental vector field associated to the Lie algebra element X.

One should now verify that ω satisfies the same Axioms for the bundle $\mathrm{Gal}_0 \times \mathbb{R} \hookrightarrow P \to \bar{\mathscr{M}}$:

(a') The condition is identical to *(a)*.

(b') One would like to establish the relation $\mathscr{R}^*_{(h,\lambda)}\omega = \mathrm{Ad}\left(h^{-1}\right)\omega$, with $h \in \mathrm{Gal}_0$ and $\lambda \in \mathbb{R}$. Using $\mathscr{R}_{(h,\lambda)} = R_h \circ \hat{\varphi}_\xi(\lambda)$, we get

$$\mathscr{R}^*_{(h,\lambda)}\omega = \hat{\varphi}_\xi(\lambda)^* R_h^*\omega = \mathrm{Ad}\left(h^{-1}\right)\hat{\varphi}_\xi(\lambda)^*\omega = \mathrm{Ad}\left(h^{-1}\right)\omega$$

where we used that $\hat{\varphi}_\xi(\lambda)$ preserves ω (*cf.* Lemma 6.3.4).

(c') This Axiom is straightforward from it (c) and $\omega\left(\hat{\xi}\right) = M$.

□

Proposition 6.4.1 then ensures that $\mathrm{Gal}_0 \times \mathbb{R} \hookrightarrow P \to \bar{\mathscr{M}}$ is a non-reductive Cartan-Bargmann geometry. Furthermore, the torsion-free condition imposed on $\mathrm{Gal}_0 \hookrightarrow P \to \mathscr{M}$ ensures in particular the vanishing of the part of the curvature 2-form Ω taking values in the mass operator M. Recalling the construction of Section 6.2, the torsion-free condition thus guarantees the involutivity of the distribution $\mathcal{D}$ defined in eq.(6.2.18), so that a reductive Cartan-Galilei geometry can be embedded inside the non-reductive Cartan-Bargmann geometry.

The sub-text which underlies the present line of reasoning is similar to the one argued in Section 3.2.2 and can be summarised by saying that a sound geometric understanding blurs the distinction between nonrelativistic structures and the gravitational waves embedding them.

Annexes

Appendix A

Mathematical background

In this Appendix, some preliminary, text-book level, material is introduced. Its ambition is by no means to provide a detailed account of the subjects covered but rather to compile some definitions and results which will reveal useful in the rest of this work.

A.1 Vector spaces

In the following, we let V and W be two vector spaces and U be a subspace of V :

Definition A.1.1 (Annihilator). *The annihilator of U in V^*, denoted by* Ann U, *is the set of linear forms $\alpha \in V^*$ which vanish on U, i.e.* $\text{Ann } U \equiv \{\alpha \in V^* \,/\, \alpha(u) = 0\,, \forall\, u \in U\}$.

Definition A.1.2 (Radical). *The radical of a bilinear form $g \in \vee^2 V^*$ is the subspace of V defined as* $\text{Rad } g \equiv \{v \in V \,/\, g\,(v, w) = 0, \,\forall\, w \in V\}$.

Proposition A.1.3. *Let $\sim$ be an equivalence relation on V. We denote $V/\sim$ the quotient space of V by $\sim$ and π the quotient map $\pi : V \to V/\sim$. If $g : V \to W$ is a continuous map such that $a \sim b$ implies $g(a) = g(b)$ for all a and b in V, then there exists a unique continuous map $f : V/\sim \to W$ such that $g = f \circ \pi$. We say that g descends to the quotient.*

Proposition A.1.4. *The quotient V/U is also a vector space and there exists a natural isomorphism between $(V/U)^*$ and* Ann U *the annihilator of U in V^* i.e.* $(V/U)^* \simeq \text{Ann } U$.

Proposition A.1.5. *Let $g \in \vee^2 V^*$ be a degenerate bilinear form with radical* Rad g. *Let us denote by Q the quotient space $Q \equiv V/\text{Rad}\, g$, with quotient map $\pi : V \to Q$. The bilinear map $\bar{g} \in \vee^2 Q^*$ defined as*

$$\bar{g}\,(\bar{v}, \bar{w}) = g\,(v, w) \tag{A.1.1}$$

with $\bar{v}, \bar{w} \in Q$ and $v \in \pi^{-1}(\bar{v})$, $w \in \pi^{-1}(\bar{w})$ is a well-defined nondegenerate bilinear form acting on Q.

Definition A.1.6 (Transpose of a linear map). *Let $f : V \to W$ be a linear map. The transpose of f, denoted $\bar{f}$ is the linear map $\bar{f} : W^* \to V^*$ defined by its action on $X \in V$ and $\alpha \in W^*$ as*

$$\bar{f}(\alpha)(X) = \alpha(f(X)). \tag{A.1.2}$$

A.2 Maps

Definition A.2.1 (Pushforward of tangent vectors). *Let $\varphi : \mathscr{M} \to \mathcal{N}$ be a smooth map. The (pointwise)-pushforward of φ at $m \in \mathscr{M}$ is the map $\varphi_* : T_m\mathscr{M} \to T_{\varphi(m)}\mathcal{N}$ defined by*

$$(\varphi_* X_m)[f] = X_m [f \circ \varphi]. \tag{A.2.3}$$

with $X_m \in T_m\mathscr{M}$ and $f \in C^\infty(\mathcal{N})$.

So, given a vector field $X \in \Gamma(T\mathscr{M})$, for each point $m \in \mathscr{M}$, one obtains a vector $\varphi_* X_m \in T_{\varphi(m)}\mathcal{N}$. However, in the general case, this does not yield a vector field on $\mathcal{N}$ *i.e.* the pushforward does not map vector fields of $\mathscr{M}$ to vector fields of $\mathcal{N}$. Two problems can occur:

- If φ is not surjective, one cannot define a vector at the points $n \in \mathcal{N} \smallsetminus \varphi(\mathscr{M})$.
- If φ is not injective, there may be more than one vectors defined at the same point of $\varphi(\mathscr{M})$.

The following Definition and Proposition give a precise meaning to the notion of vector fields related by pushforward, as well as an equivalent, in this case, of equation (A.2.3).

Definition A.2.2 (Related vector fields). *Let $\varphi : \mathscr{M} \to \mathcal{N}$ be a smooth function and $X \in \Gamma(T\mathscr{M})$ be a vector field on $\mathscr{M}$. If a vector field $Y \in \Gamma(T\mathcal{N})$ on $\mathcal{N}$ satisfies $\varphi_{m*} X_m = Y_{\varphi(m)}$ $\forall m \in \mathscr{M}$, then X and Y are said to be φ-related.*

Proposition A.2.3. *Let $\varphi : \mathscr{M} \to \mathcal{N}$ be a smooth function. The vector fields $X \in \Gamma(T\mathscr{M})$ and $Y \in \Gamma(T\mathcal{N})$ are φ-related if and only if*

$$X[f \circ \varphi] = Y[f] \circ \varphi \tag{A.2.4}$$

for all $f \in C^\infty(N)$.

By means of the pushforward, each smooth map yields a linear map between tangent vector spaces. Dualizing leads to a linear map between cotangent vector spaces called the pullback:

Definition A.2.4 (Pullback)**.** *Let $\varphi : \mathscr{M} \to \mathcal{N}$ be a smooth map. The (pointwise)-pullback of φ at $m \in \mathscr{M}$ is the map $\varphi^* : T^*_{\varphi(m)}\mathcal{N} \to T^*_m\mathscr{M}$ defined by*

$$\varphi^*\omega_{\varphi(m)}\left(X_m\right) = \omega_{\varphi(m)}\left(\varphi_* X_m\right) \tag{A.2.5}$$

with $\omega_{\varphi(m)} \in T^*_{\varphi(m)}\mathcal{N}$ and $X_m \in T_m\mathscr{M}$.

Contrarily to the pushforward case, there is no ambiguity when passing from pointwise-pullback to the pullback of covector fields, so that the pullback always maps covector fields to covector fields:

Proposition A.2.5. *Given a smooth map $\varphi : \mathscr{M} \to \mathcal{N}$ and a covector field $\omega \in \Omega^1(\mathcal{N})$, the pullback of ω by φ defined as $\left(\varphi^*\omega\right)_m = \varphi^*\omega_{\varphi(m)}$ is a covector field.*

For covector fields, expression A.2.5 becomes:

$$\left(\varphi^*\omega\right)(X) = \omega\left(\varphi_* X\right) \circ \varphi. \tag{A.2.6}$$

Proposition A.2.6. *Let $f : \mathscr{M} \to \mathcal{N}$, $g : \mathcal{N} \to \mathcal{P}$ and $h : \mathscr{M} \to \mathcal{P}$ be smooth functions, with*

$$h = g \circ f. \tag{A.2.7}$$

The pushforward and pullbacks of the composite function h are given respectively by:

$$\begin{aligned} h_* &= g_* \circ f_* \\ h^* &= f^* \circ g^*. \end{aligned}$$

Pushforward and pullback can be generalised to arbitrary tensors as follows:

Definition A.2.7 (Generalisation of pushforward and pullback to arbitrary tensors)**.** *Suppose $\varphi : \mathscr{M} \to \mathcal{N}$ is a diffeomorphism. For any pair of nonnegative integers k, l, there are smooth isomorphisms $\varphi_* : \Gamma\left(T^{(k,l)}\mathscr{M}\right) \to \Gamma\left(T^{(k,l)}\mathcal{N}\right)$ and $\varphi^* : \Gamma\left(T^{(k,l)}\mathcal{N}\right) \to \Gamma\left(T^{(k,l)}\mathscr{M}\right)$ satisfying:*

$$\begin{aligned} \varphi_* S\left(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l\right) &= S\left({\varphi^{-1}}_* X_1, \ldots, {\varphi^{-1}}_* X_k, \varphi^*\omega^1, \ldots, \varphi^*\omega^l\right) \\ \varphi^* T\left(Y_1, \ldots, Y_k, \chi^1, \ldots, \chi^l\right) &= T\left(\varphi_* Y_1, \ldots, \varphi_* Y_k, \varphi^{-1*}\chi^1, \ldots, \varphi^{-1*}\chi^l\right) \end{aligned}$$

with $X_1, \ldots, X_k \in \Gamma(T\mathcal{N})$, $Y_1, \ldots, Y_k \in \Gamma(T\mathscr{M})$, $\omega^1, \ldots, \omega^l \in \Omega^1(\mathcal{N})$, $\chi^1, \ldots, \chi^l \in \Omega^1(\mathscr{M})$, $S \in \Gamma\left(T^{(k,l)}\mathscr{M}\right)$ and $T \in \Gamma\left(T^{(k,l)}\mathcal{N}\right)$.

We conclude this Section by defining

Definition A.2.8 (Submersion)**.** *A smooth map $\pi : \mathscr{M} \to \mathcal{N}$ is called a submersion if the pushforward $\pi_* : T_m\mathscr{M} \to T_{i(m)}\mathcal{N}$ is surjective at each point.*

Definition A.2.9 (Immersion)**.** *A smooth map $i : \mathscr{M} \to \mathcal{N}$ is called an immersion if the pushforward $i_* : T_m\mathscr{M} \to T_{i(m)}\mathcal{N}$ is injective at each point.*

Proposition A.2.10. *Let $f : \mathscr{M} \to \mathcal{N}$ be a submersion, then the pullback $f^* : T^*\mathscr{M} \to T^*\mathcal{N}$ is injective at each point.*

A.3 Distributions

Definition A.3.1 (Distribution)**.** *A distribution of rank r on $\mathscr{M}$ is a collection $\mathcal{D} = \{\mathcal{D}_x\}$ of r-dimensional subspaces $\mathcal{D}_x \subset T_x\mathscr{M}$, one for each $x \in \mathscr{M}$.*

In the language of vector bundles (*cf.* Section A.6), a distribution can be thought of as subbundle of the tangent bundle.

Notation A.3.2. Let $X \in \Gamma(T\mathscr{M})$ be a vector field on $\mathscr{M}$ and $\mathcal{D} = \{\mathcal{D}_x\}$ a distribution on $\mathscr{M}$, the notation $X \in \mathcal{D}$ stands for $X_x \in \mathcal{D}_x$, $\forall x \in \mathscr{M}$.

Definition A.3.3 (Integral manifold)**.** *Let $i : \mathscr{M}' \to \mathscr{M}$ be an immersion of $\mathscr{M}'$ in $\mathscr{M}$. Suppose $\mathcal{D}$ is a distribution on $\mathscr{M}$. Then $\mathscr{M}'$ is called an integral manifold of $\mathcal{D}$ if for any $x' \in \mathscr{M}'$ we have $(i_*)_{x'}(T_{x'}\mathscr{M}') = \mathcal{D}_{i(x')}$.*

Definition A.3.4 (Involutive distribution)**.** *A distribution $\mathcal{D}$ on $\mathscr{M}$ is said involutive if for all vector fields $X, Y \in \Gamma(T\mathscr{M})$ such that $X, Y \in \mathcal{D}$, one has $[X, Y] \in \mathcal{D}$ where $[\cdot, \cdot]$ stands for the Lie bracket of vector fields.*

From now on, we will focus on the case where the subspaces $\mathcal{D}_x \in T_x\mathscr{M}$ are hypersurfaces of codimension 1.

Proposition A.3.5 (Frobenius Criterion)**.** *For the special case of a distribution $\mathcal{D}$ defined as the kernel of a 1-form $\alpha \in \Omega^1(\mathscr{M})$ i.e. $\mathcal{D}_x = \mathrm{Ker}\,\alpha_x = \{X_x \in T_x\mathscr{M} / \alpha_x(X_x) = 0\}$, $\mathcal{D}$ is involutive if and only if $\alpha(X) = \alpha(Y) = 0 \Rightarrow d\alpha(X, Y) = 0$. Equivalently, $\mathcal{D}$ is involutive if and only if $\alpha \wedge d\alpha = 0$.*

Proof: According to Definition A.3.4, the distribution $\mathcal{D}$ is involutive if $\forall X, Y \in \Gamma(T\mathscr{M})$ such that $\alpha(X) = \alpha(Y) = 0$ we have $\alpha([X, Y]) = 0$. According to the definition of the exterior derivative of a 1-form, we have $d\alpha(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y]) = -\alpha([X, Y])$ and thus $d\alpha(X, Y)$ must vanish for $X, Y \in \mathcal{D}$. □

Definition A.3.6 (Maximal integral submanifold). *Let $\mathcal{D}$ be a distribution on a manifold $\mathscr{M}$ of dimension n defined as the kernel of a 1-form $\alpha \in \Omega^1(\mathscr{M})$ and $i : \mathscr{M}' \to \mathscr{M}$ be an immersion of $\mathscr{M}'$ in $\mathscr{M}$ such that $\mathscr{M}'$ is an integral manifold of $\mathscr{M}$. $\mathscr{M}'$ is said maximal if $\operatorname{Ker} i^* = \operatorname{Span} \alpha$ and $\mathscr{M}'$ is then of dimension $n-1$.*

Theorem A.3.7 (Frobenius Theorem). *A distribution $\mathcal{D}$ on $\mathscr{M}$ is integrable, if and only if, it is involutive. Moreover, through every point $x \in \mathscr{M}$ there passes a unique maximal integral manifold of $\mathcal{D}$ and every other integral manifold containing x is an open submanifold of the maximal one.*

A.4 Group action

Definition A.4.1 (Left action). *A left action of a Lie group G on a manifold $\mathscr{M}$ is a mapping $\Phi : G \times \mathscr{M} \to \mathscr{M} : (g, m) \mapsto \Phi(g, m) \equiv g \cdot m$ such that :*
- *Φ is associative: $\forall g, h \in G$, and $m \in \mathscr{M}$, $g \cdot (h \cdot m) = (gh) \cdot m$.*
- *$e \cdot m = m$, $\forall m \in \mathscr{M}$ with e the identity element of G.*

Definition A.4.2 (Right action). *A right action of a Lie group G on a manifold $\mathscr{M}$ is a mapping $\rho : \mathscr{M} \times G \to \mathscr{M} : (m, g) \mapsto \rho(m, g) \equiv m \cdot g$ such that :*
- *ρ is associative: $\forall g, h \in G$, and $m \in \mathscr{M}$, $(m \cdot g) \cdot h = m \cdot (gh)$.*
- *$m \cdot e = m$, $\forall m \in \mathscr{M}$ with e the identity element of G.*

Definition A.4.3 (G-space). *Let S be a non-empty set and G a Lie group. Then S is called a G-space if it is equipped with an action of G on S.*

Definition A.4.4 (G-space isomorphism). *Let S_1 and S_2 be two G-spaces with G-actions $\Phi_1 : G \times S_1 \to S_1 : (g, x_1) \mapsto g \cdot x_1$ and $\Phi_2 : G \times S_2 \to S_2 : (g, x_2) \mapsto g \bullet x_2$ respectively. The sets S_1 and S_2 are said G-space isomorphic if there exists a bijective map $f : S_1 \to S_2$ such that $f(g \cdot x_1) = g \bullet f(x_1)$ for all $x_1 \in S_1$ and $g \in G$. The map f is then called a G-space isomorphism.*

Definition A.4.5 (Orbit). *The set $G \cdot m = \{\Phi(g, m) \in \mathscr{M} \,|\, g \in G\}$ is called the orbit of the G-action Φ through m.*

Definition A.4.6 (Isotropy group). *The set $G_m = \{g \in G \,|\, \Phi(g, m) = m\}$ is the isotropy group at m.*

Definition A.4.7 (Orbit space). *The set $\mathscr{M}/G \equiv \{G \cdot m \,|\, m \in \mathscr{M}\}$ of all G-orbits on $\mathscr{M}$ is called the orbit space of the action Φ.*

Definition A.4.8 (Effective action). *An action $\Phi : G \times \mathscr{M} \to \mathscr{M}$ is said effective if and only if $\forall g \neq e \in G$, $\exists m \in \mathscr{M} \,|\, \Phi(g, m) \neq m$.*

Definition A.4.9 (Free action)**.** *An action Φ is said free if and only if $\forall m \in \mathscr{M}$, $\Phi(g, m) = m$ implies $g = e$. In other words, the isotropy groups of a free action at each point of m contain only the identity element of G.*

Note that a free action is necessarily effective.

Definition A.4.10 (Transitive action)**.** *An action $\Phi : G \times \mathcal{N} \to \mathcal{N}$ will be said transitive on $\mathcal{N} \subset \mathscr{M}$ if and only if for any $m_1, m_2 \in \mathcal{N}$, $\exists\, g \in G \,|\, \Phi(g, m_1) = m_2$.*

Proposition A.4.11. *The action Φ is transitive inside each orbit $G \cdot m$.*

> **Proof:** Given an orbit $G \cdot m$, the action will be said transitive if for any couple $\varphi(g_1, m), \varphi(g_2, m) \in G \cdot m$, there exists $g \in G$ such that $\varphi(g_1, m) = \varphi(g, \varphi(g_2, m))$. The associativity property ensures that $\varphi(g, \varphi(g_2, m)) = \varphi(gg_2, m)$ and then $g = g_1 g_2^{-1}$. $\square$

Definition A.4.12 (Regular action)**.** *A transitive and free action is said regular. For any two $m_1, m_2 \in \mathscr{M}$, there exists precisely one $g \in G$ such that $g.m_1 = m_2$.*

Definition A.4.13 (Homogeneous manifold)**.** *If Φ is transitive globally on $\mathscr{M}$, then every point of $\mathscr{M}$ belong to the same orbit and $\mathscr{M}$ is said homogeneous. The orbit space then reduces to one point.*

Definition A.4.14 (Proper manifold)**.** *A G-action Φ acting on a smooth manifold $\mathscr{M}$ is proper if and only if the map $\Xi : G \times \mathscr{M} \to \mathscr{M} \times \mathscr{M} : (g, m) \mapsto (m, \Phi(g, m))$ is proper i.e. for every compact set $K \subset \mathscr{M} \times \mathscr{M}$, the inverse image $\Xi^{-1}(K)$ is compact.*

Definition A.4.15 (Coset space)**.** *Let G be a Lie group and $H \subset G$ a Lie subgroup. A coset, or left lateral class, gH is defined as $gH = \{gh \,/\, h \in H\}$. We denote by G/H the coset space of all lateral classes gH i.e. $G/H = \{gH \,/\, g \in G\}$.*

Proposition A.4.16. *Let G be a Lie group and $H \subset G$ a Lie subgroup. Furthermore, let $g, g' \in G$. The cosets gH and $g'H$ are equal if and only if $g^{-1}g \in H$.*

> **Proof:** We want to prove the equivalence
>
> $$g^{-1}g' \in H \Leftrightarrow g'H = gH$$
>
> which can be restated as
>
> $$\exists h \in H/g' = gh \Leftrightarrow \forall h' \in H, \exists h \in H/g'h' = gh.$$

We first establish the sufficiency and then the necessity:

$\Longrightarrow$:
Since $g' = gh$ one immediatly obtains $g'H = ghH = gH$.

$\Longleftarrow$:
Choosing $h' = e$, $\exists h / g' = gh$. $\square$

A.5 Lie groups

Let G be a Lie group.

Definition A.5.1 (Left translation)**.** *To each element $g \in G$ corresponds a left translation map defined as the map $L_g : G \to G : h \mapsto gh$.*

Definition A.5.2 (Left-invariant vector fields)**.** *A vector field $v \in \Gamma(TG)$ is left-invariant if $L_{g*} v_h = v_{gh}$.*

Proposition A.5.3. *There is a one-to-one correspondence between the set of left-invariant vector fields and the set of vectors tangent to G at e.*

Proposition A.5.4. *The set of left-invariant vector fields is closed under the Lie bracket operation.*

According to Theorem A.5.3, the space of left-invariant vector fields on a Lie group G is a vector space. Besides, it is endowed with a bracket and then defines a Lie algebra denoted $\mathfrak{g}$. Elements of $\mathfrak{g}$ are then in one-to-one correspondence with elements of T_eG and there is a canonical isomorphism $\phi : T_eG \mapsto \mathfrak{g}$. One should be careful that altough the isomorphism ϕ preserves the vector space structure of $\mathfrak{g}$, it does not preserve the Lie algebra bracket.

Definition A.5.5 (Adjoint representation)**.** *The adjoint representation is a representation of a Lie group G on its Lie algebra $\mathfrak{g}$. Consider the map $L_g R_{g^{-1}} : G \mapsto G : h \mapsto ghg^{-1}$ and its associated pushforward $\left(L_g R_{g^{-1}}\right)_* : T_hG \mapsto T_{ghg^{-1}}G$. At $h = e$, $\left(L_g R_{g^{-1}}\right)_*\big|_e : T_eG \mapsto T_eG$. Using the isomorphism $\phi : T_eG \mapsto \mathfrak{g}$, it defines an application from $\mathfrak{g}$ to $\mathfrak{g}$ $\mathrm{Ad}(g) = \phi \circ \left(L_g R_{g^{-1}}\right)_*\big|_e \circ \phi^{-1} : \mathfrak{g} \mapsto \mathfrak{g}$. Furthermore, the map $\mathrm{Ad} : g \mapsto \mathrm{Ad}(g)$ is a representation.*

Definition A.5.6 (One-parameter subgroup)**.** *Let G be a Lie group. A curve $\gamma : \mathbb{R} \mapsto G$ is called a one-parameter subgroup of G if it satisfies the condition $\gamma(t)\gamma(s) = \gamma(t+s)$. It then defines an abelian subgroup of G.*

Proposition A.5.7. *There is a one-to-one correspondence between a one-parameter subgroup of a Lie group G and a left-invariant vector field on G.*

Definition A.5.8 (Exponential map). *Let G be a Lie group and $V \in T_eG$. The exponential map* $\exp : T_eG \mapsto G$ *is defined by* $\exp V \equiv \gamma_V(1)$ *where γ_V is the one-parameter subgroup of G generated by the left-invariant vector field $X_V|_g = L_{g*}V$.*

Proposition A.5.9. *Given $V \in T_eG$, $t \in \mathbb{R}$, the exponential map satisfies* $\exp(tV) = \gamma_V(t)$.

Definition A.5.10 (Fundamental vector field). *Let G be a Lie group with associated Lie algebra $\mathfrak{g}$. Let $\mathscr{M}$ be a smooth manifold endowed with a smooth action $\Phi : G \times \mathscr{M} \mapsto \mathscr{M}$. For $X \in \mathfrak{g}$, we define the fundamental vector field $X^\sharp$ on $\mathscr{M}$ as $X^\sharp_p(f) = \frac{d}{dt}\big|_{t=0} f(\Phi(\exp(tX), p))$, for $f \in C^\infty(\mathscr{M})$. The map $\sigma : \mathfrak{g} \mapsto \Gamma(T\mathscr{M}) : X \mapsto X^\sharp$ is a Lie algebra homomorphism, i.e. $\left[X^\sharp, Y^\sharp\right] = [X,Y]^\sharp$ for $X, Y \in \mathfrak{g}$. If G acts effectively on $\mathscr{M}$, then σ is an isomorphism. If G acts freely on $\mathscr{M}$, then for each non-zero $X \in \mathfrak{g}$, $\sigma(X)$ never vanishes on $\mathscr{M}$.*
For future applications, we consider the case where Φ is the right action $R : P \times G \mapsto P$ of a Lie group G acting on a principal fiber bundle P. The action of $X^\sharp_p$ on a function $f \in C^\infty(P)$ reads $X^\sharp_p(f) = \frac{d}{dt}\big|_{t=0} f(R(p, \exp(tX)))$.

Proposition A.5.11. *Let $\pi : \mathscr{M} \to \mathscr{M}$ be a diffeomorphism. If a vector field X generates a flow $\varphi_t : \mathscr{M} \to \mathscr{M}$, the vector field $\pi_* X$ generates the flow $\phi_t \equiv \pi \circ \varphi_t \circ \pi^{-1}$.*

Proposition A.5.12. *Let $X^\sharp$ be the fundamental vector field corresponding to the Lie algebra element $X \in \mathfrak{g}$. For each $a \in G$, $R_{a*}X^\sharp$ is the fundamental vector field corresponding to the Lie algebra element* $\mathrm{Ad}\left(a^{-1}\right) X \in \mathfrak{g}$, *i.e.* $\sigma\left(\mathrm{Ad}\left(a^{-1}\right) X\right) = R_{a*}X^\sharp$.

A.6 Fiber bundle

A (differentiable) fiber bundle $(E, \pi, \mathscr{M}, F, G)$ consists of the following elements:

1. A differentiable manifold E called the total space.
2. A differentiable manifold $\mathscr{M}$ called the base space.
3. A differentiable manifold F called the fiber (or typical fiber).
4. A surjection $\pi : E \to \mathscr{M}$ called the projection. The inverse image $\pi^{-1}(p) = F_p \cong F$ is called the fiber at p.
5. A Lie group G called the structure group, which acts on F on the left.
6. A set of open covering $\{U_i\}$ of $\mathscr{M}$ with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The map ϕ_i is called the local trivialisation since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$.

7. If we write $\phi_i(p, f) = \phi_{i,p}(f)$, the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism. On $U_i \cap U_j \neq \emptyset$, we require that $t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ be an element of G. Then ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f). \tag{A.6.8}$$

The maps t_{ij} are called the transition functions.

Definition A.6.1 (Principal bundle). *A principal bundle has a fiber F which is identical to the structure group G. A principal bundle $P \xrightarrow{\pi} \mathscr{M}$ is also denoted by $P(\mathscr{M}, G)$ and is often called a G-bundle over $\mathscr{M}$. In addition to the usual left action of G on the fiber, we can define a right action of G on F as follows: Let $\phi_i : U_i \times F \mapsto \pi^{-1}(U_i)$ be the local trivialisation given by $\phi^{-1}(u) = (p, g_i)$, where $u \in \pi^{-1}(U_i)$ and $p = \pi(u)$. The right action $R : P \times G \mapsto P$ is defined by*

$$ua \equiv R(u, a) = \phi_i(p, g_i a) \tag{A.6.9}$$

$\forall a \in G$ and $u \in \pi^{-1}(p)$. The right action of G on P is free and proper and is furthermore fiber preserving and transitive on each fiber.

Theorem A.6.2 (Quotient manifold Theorem). *Let $\mathscr{M}$ be a smooth manifold on which acts the G-action Φ. If Φ is free and proper, then the orbit space $\mathscr{M}/G$ has a unique smooth manifold structure such that the projection map $\pi : \mathscr{M} \to \mathscr{M}/G = \bar{\mathscr{M}} : m \mapsto G \cdot m$ is a smooth submersion and defines a principal fiber bundle with structure group G. A fiber of π is a G-orbit in $\mathscr{M}$ and is then diffeomorphic to the structure group G with diffeomorphism $G \cdot m \to G : \Phi(g, m) \mapsto g$.*

Definition A.6.3 (Local section). *A fiber bundle admits a local section if for every $\bar{m} \in \bar{\mathscr{M}}$ there exists an open neighborhood $\bar{U}_i$ of m and a smooth mapping $\sigma_i : \bar{U}_i \subseteq \bar{\mathscr{M}} \to \pi^{-1}(\bar{U}_i)$ such that $\pi \circ \sigma_i = id_{\bar{U}_i}$.*

Definition A.6.4 (Local trivialisation). *A local trivialisation is a set of open covering $\{\bar{U}_i\}$ of $\bar{\mathscr{M}}$ with a diffeomorphism $\phi_i : \bar{U}_i \times G \to \pi^{-1}(\bar{U}_i)$ such that $\pi \circ \phi_i(\bar{u}, g) = \bar{u}$.*

Proposition A.6.5. *Given a local section $(\{\bar{U}_i\}, \sigma_i)$, the mapping $\phi_{\sigma_i} : \bar{U}_i \times G \to \pi^{-1}(\bar{U}_i) : (\bar{u}, g) \mapsto \Phi(g, \sigma_i(\bar{u}))$ defines a local trivialisation of the fibration π. In this local trivialisation, the section $\sigma_i(\bar{u})$ is expressed as $\sigma_i(\bar{u}) = \phi_{\sigma_i}(\bar{u}, e)$.*

Proof: Since we quotiented the manifold $\mathscr{M}$ by the action of Φ, the map $\Phi : G \times \pi^{-1}(\bar{u}) \to \pi^{-1}(\bar{u})$ preserves the fiber $\pi^{-1}(\bar{u})$ $\forall \bar{u} \in \bar{U}_i$ such that $\pi \circ \phi_{\sigma_i} = id_{\bar{U}_i}$. Besides, the action of Φ inside a fiber is regular, being both transitive and free and then ϕ_{σ_i} is an isomorphism. Since we only consider smooth functions, it is also a diffeomorphism. □

Proposition A.6.6. *A principal bundle is trivial if and only if it admits a global cross section.*

Proof: Since local cross sections are in one-to-one correspondence with local trivialisations, the existence of a global cross section is a necessary and sufficient condition for the existence of a global trivialisation *i.e.* the principal bundle is trivial. □

Proposition A.6.7. *A principal bundle with structure group* $(\mathbb{R}^n, +)$ *can be made trivial.*

Proof: See Proposition 16.14.5 of [155]. □

Theorem A.6.8 (Immersed submanifolds)**.** *Immersed submanifolds are precisely the images of injective immersions.*

Associated bundle

Definition A.6.9 (Tensor)**.** *Let* V *be a vector space and* $\rho : H \to GL(V)$ *a representation. We call a tensor of type* (V, ρ) *a map* $f : P \to V$ *transforming under the right action* R_h *on* P *as:* $R_h f = \rho\left(h^{-1}\right) f$*. The set of tensors of type* (V, ρ) *is denoted* $\mathscr{T}(V, \rho)$*.*

Proposition A.6.10. *Let* V *be a vector space and* $\rho : H \to GL(V)$ *a representation on* V*. Furthermore, let* λ *be an element of* V *and define the constant function on the principal bundle* P *taking values in* V *as* $f : P \to V$*,* $f(p) = \lambda$*,* $\forall p \in P$*. Then, the function* f *is a tensor of type* (V, ρ) *if and only if* λ *is* ρ*-invariant.*

Proof: According to Definition A.6.9, the necessary and sufficient condition for f to be a tensor of type (V, ρ) is for f to satisfy the condition $f(ph) = \rho\left(h^{-1}\right) f(p)$, $\forall p \in P$ and $\forall h \in H$. Replacing f by its expression leads to $\rho\left(h^{-1}\right)\lambda = \lambda$, $\forall h \in H$, so that λ must be ρ-invariant. □

Definition A.6.11 (Associated bundle)**.** *Let* $P \to \mathscr{M}$ *be a principal bundle with structure group* H *and* F *a manifold on which* H *acts on the left through the action* $\rho : H \times F \to F$*. Using the right-action* $R : P \times H \to P$ *of* H *on* P*, we can define the right-action* $\tilde{R} : (P \times F, H) \to P \times F$ *of* H *on the product manifold* $P \times F$ *as follows: an element* $h \in H$ *maps* $(p, \xi) \mapsto \left(R_h p, \rho\left(h^{-1}\right)\xi\right)$*. The quotient space of* $P \times F$ *by this right action is denoted* $E = P \times_H F$ *and is called the associated bundle of* P *with standard fiber* F*.*

The mapping $P\times F \to \mathscr{M}$ which maps (p,ξ) into $\pi(p)$ induces a mapping $\pi_E : E \to \mathscr{M}$ called the projection of E onto $\mathscr{M}$. We denote Φ the natural projection map $\Phi : P\times F \to E$ which sends $(p,\xi) \in P\times F$ to the associated equivalence class denoted $\Phi(p,\xi) \in \pi_E^{-1}(x) \subset E$, where $\pi(p) = x$.

$$\begin{tikzcd} P\times F \arrow[r, "\Phi"] \arrow[d, "p_r"] & E \arrow[d, "\pi_E"] \\ P \arrow[r, "\pi"] & \mathscr{M} \end{tikzcd}$$

Each point $p\in P$ can then be seen as a bijective map $p : F \to \pi_E^{-1}(x) : \xi \mapsto \Phi(p,\xi)$, where again $\pi(p) = x$. We have the following relation:

$$\Phi(R_h p,\xi) = \Phi(p,\rho(h)\xi) \tag{A.6.10}$$

with $h\in H$, $p\in P$ and $\xi\in F$ which comes from the fact that the elements $(R_h p,\xi)$ and $\tilde{R}_{h^{-1}}(R_h p,\xi) = (p,\rho(h)\xi)$ are by definition in the same equivalence class. This property is useful in order to prove the following Proposition:

Proposition A.6.12. *The space of functions $\mathscr{T}(F,\rho)$ is isomorphic to the space of sections on E.*

Proof: We consider the following commuting diagram:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"] & F \arrow[d, "p"] \\ \mathscr{M} \arrow[r, "\sigma"] & E. \end{tikzcd}$$

We show first that the gift of a section $\sigma : \mathscr{M} \to E$ induces a function $f \in \mathscr{T}(F,\rho)$. The map $p : F \to \pi_E^{-1}(x)$ being inversible, we define the function $f : P \to F$ as $f(p) = p^{-1}(\sigma(\pi(p)))$. Using the following Lemma:

Lemma A.6.13. $(R_h p)^{-1}(e) = \rho\left(h^{-1}\right)p^{-1}(e)$ *with* $e\in E$.

which follows readily from relation (A.6.10), we have $f(R_h p) = (R_h p)^{-1}(\sigma(\pi(R_h p))) = (R_h p)^{-1}(\sigma(\pi(p))) = (R_h p)^{-1}(p(f(p))) = \rho\left(h^{-1}\right)f(p)$ so that $f\in\mathscr{T}(F,\rho)$.

Conversely, one can define a section $\sigma : \mathscr{M} \to E$ starting from a function $f \in \mathscr{T}(F,\rho)$ as $\sigma(x) = p(f(p))$ with $p\in P$ such that $\pi(p) = x$. We start by showing that this definition is independent of the choice of p. Starting from a different representative p', there exists an element $h\in H$ such that $p' = R_h p$ and we have: $p'(f(p')) = R_h p(f(R_h p)) = R_h p\left(\rho\left(h^{-1}\right)f(p)\right) = p(f(p))$. Secondly, we check that σ is indeed a section *i.e.* that $\pi_E \circ \sigma = I_d(\mathscr{M})$. This is done as follows: $\pi_E(\sigma(x)) = \pi_E(p(f(p))) = \pi(p) = x$. $\square$

A.7 Connections

Definition A.7.1 (Ehresmann connection). *Let P be a principal bundle over the base $\mathscr{M}$ with fiber $F \simeq G$. The associated algebra of the Lie group G is denoted $\mathfrak{g}$ and we call $R : P \times G \mapsto P$ the right action of G on P, which is free and proper. The right action, being free, defines an isomorphism between elements of $\mathfrak{g}$ and fundamental vector fields on P via the map $\sigma : \mathfrak{g} \to \Gamma(TP) : X \mapsto X^\sharp$ where the action of $X^\sharp$ on a function $f \in \mathcal{F}(P)$ reads $X^\sharp_p(f) = \frac{d}{dt}\big|_{t=0} f(R(p, \exp(tX)))$. Being free, the action R is also effective therefore the vector field $\sigma(X)$ never vanishes on P for each non-zero $X \in \mathfrak{g}$. The dim (G)-dimensional vector subspace $V_pP \subset T_pP$ spanned by all $X^\sharp_p$ is called vertical subspace and is isomorphic to $\mathfrak{g}$ via σ.*

Proposition A.7.2. *$VP = Ker(\pi_*)$.*

Proof: For any $f \in C^\infty(\mathscr{M})$ and $X \in \mathfrak{g}$, we have:

$$\begin{aligned}
\pi_* X^\sharp(f) &= X^\sharp(f \circ \pi) \\
&= \frac{d}{d\lambda}\left[f \circ \pi(R(p, exp(\lambda X)))\right]\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda}(f \circ \pi(p))\Big|_{\lambda=0} \\
&= 0
\end{aligned}$$

where $\pi(R(g,p)) = \pi(p)$ was used.

So $VP \subset Ker(\pi_*)$. According to the rank-nullity Theorem, $dim(Im\,\pi_*) + dim(Ker\,\pi_*) = dim\,TP$. Using, $dim\,TP = dim\,G + dim\,\mathscr{M}$ and that π is a submersion, therefore $dim(Im\,\pi_*) = dim\,\mathscr{M}$, we obtain that $dim(Ker\,\pi_*) = dim\,G = dim\,VP$. Therefore, $VP = Ker(\pi_*)$. $\square$

The supplementary of VP in TP, called horizontal space is not uniquely defined.

Definition A.7.3 (Ehresmann connection). *An Ehresmann connection is the gift of an equivariant distribution of horizontal spaces:*

$$TP = VP \oplus HP \tag{A.7.11}$$

such that $H_{pg} = R_{g*}H_p$.

Proposition A.7.4. *π_* is an isomorphism between H_p and $T_{\pi(p)}\mathscr{M}$.*

Proof: H_p and $T_{\pi(p)}\mathscr{M}$ share the same dimension and are therefore isomorphic. Plus, $\pi_* : H_p \subset T_pP \to T_{\pi(p)}\mathscr{M}$ is surjective (π being a submersion) and then is an isomorphism. $\square$

The preceding Definition of an Ehresmann connection is equivalent to the gift of a 1-form $\omega_p \in T^*P_p \otimes \mathfrak{g}$ such that $Ker\,\omega_p = H_p$ and $\omega_p\left(X_p^\sharp\right) = X$ with $X_p^\sharp \in V_p$ and $X \in \mathfrak{g}$ that is, ω_p acting on V_pP is the inverse isomorphism of $\sigma_p : \mathfrak{g} \to V_pP$.

Proposition A.7.5. *The 1-form ω is equivariant, i.e. $\omega_{R(p,g)}\left(R_{g*}\left(X_p\right)\right) = \mathrm{Ad}\left(g^{-1}\right)\omega_p\left(X_p\right)$, $\forall X_p \in T_pP$.*

Proof: We decompose X_p in vertical and horizontal parts as $X_p = v_p^\sharp + h_p$, with $v_p^\sharp \in V_p$ and $h_p \in H_p$. According to Proposition A.5.12, a fundamental vector field is equivariant, *i.e.* $R_{g*}v_p^\sharp = \sigma_{R(p,g)}\left(\mathrm{Ad}\left(g^{-1}\right)v\right)$. Applying $\omega_{R(p,g)}$:

$$\begin{aligned}
\omega_{R(p,g)}\left(R_{g*}v_p^\sharp\right) &= \omega_{R(p,g)}\left(\sigma_{R(p,g)}\left(\mathrm{Ad}\left(g^{-1}\right)v\right)\right)\\
&= \mathrm{Ad}\left(g^{-1}\right)v\\
\omega_{R(p,g)}\left(R_{g*}v_p^\sharp\right) &= \mathrm{Ad}\left(g^{-1}\right)\omega_p\left(v_p^\sharp\right).
\end{aligned}$$

therefore the Proposition is satisfied for vertical vectors.

For the horizontal part, according to the equivariance of H_p, $R_{g*}h_p \in H_{R(p,g)}$, therefore $\omega_{R(p,g)}\left(R_{g*}h_p\right) = 0 = \omega_p\left(h_p\right)$. $\square$

A.8 Homomorphisms

Definition A.8.1 (Group homomorphism)**.** *A group homomorphism is a map $\rho : G \to H$ between two groups G and H such that, $\forall g_1, g_2 \in G$, the relation $\rho\left(g_1g_2\right) = \rho\left(g_1\right)\rho\left(g_2\right)$ holds.*

Definition A.8.2 (Lie group homomorphism)**.** *A Lie group homomorphism is a map $\rho : G \to H$ between two Lie groups G and H such that ρ is both a group homomorphism and a smooth map.*

Definition A.8.3 (Homomorphism of principal bundles)**.** *Let P_1 and P_2 be two principal bundles over a manifold $\mathscr{M}$ and we denote by G_1 and G_2 their respective structure groups. Let $\rho : G_1 \to G_2$ be a Lie group homomorphism. We will call the differentiable map*

$f : P_1 \to P_2$ a ρ-homomorphism between P_1 and P_2 if f is G_1-equivariant i.e. $f(p_1 g) = f(p_1)\rho(g)$, $\forall p_1 \in P_1$ and $g \in G_1$. If $P_1 = P_2$, the equivariant map $f : P \to P$ is called a bundle automorphism of P.

Proposition A.8.4. *Let $f : P_1 \to P_2$ be a ρ-homomorphism between P_1 and P_2, with bases $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively. There is a canonical map $\phi : \mathscr{M}_1 \to \mathscr{M}_2$.*

Proof: Let us denote R_1, R_2 the two right-actions and $\sigma : \mathscr{M}_1 \to P_1$ a section. Let us show that the map $\phi : \mathscr{M}_1 \to \mathscr{M}_2 : x \mapsto \pi_2 \circ f \circ \sigma(x)$ is independent of the choice of section, by introducing the section $\sigma' : \mathscr{M}_1 \to P_1$ defined as $\sigma(x) = R_1(\sigma'(x), h(x))$ with $h : P_1 \to H$ a map. We then have

$$\begin{aligned}
\phi(x) &= \pi_2 \circ f \circ \sigma(x) \\
&= \pi_2 \circ f \circ R_1\left(\sigma'(x), h(x)\right) \\
&= \pi_2 \circ R_2\left(f\left(\sigma'(x)\right), \rho(h(x))\right) \\
&= \pi_2 \circ f \circ \sigma'(x).
\end{aligned}$$

□

Definition A.8.5 (Imbedding)**.** *A ρ-homomorphism $f : P_1 \to P_2$ of principal bundles such that*

1. *$f : P_1 \to P_2$ is an injective immersion*
2. *$\rho : G_1 \to G_2$ is injective*

is called an imbedding.

The canonical map $\phi : \mathscr{M}_1 \to \mathscr{M}_2$ is then also an injective immersion.

Definition A.8.6 (Reduction of structure group)**.** *A ρ-imbedding $f : P_1 \to P_2$ between two principal bundles with bases $\mathscr{M}_1$ and $\mathscr{M}_2$ and structure groups G_1 and G_2 will be called a reduction of structure group if*

1. *G_1 is a subgroup of G_2 and $\rho : G_1 \to G_2$ is the inclusion homomorphism*
2. *$\mathscr{M}_1 = \mathscr{M}_2$ and the canonical map $\phi : \mathscr{M}_1 \to \mathscr{M}_2$ is the identity transformation.*

The homomorphism f is then called a reduction of structure group from the G_2-bundle P_2 to the G_1-bundle P_1 and P_1 is said a reduced subbundle of P_2.

Definition A.8.7 (Reduction induced by a distribution)**.** *Let P be a principal bundle and $\mathcal{D} : P \to TP$ an involutive distribution on P. Assume futhermore that P admits a reduction to a subbundle P'. We say that the reduction $P \to P'$ is induced by the distribution $\mathcal{D}$ if P' is a maximal integral manifold of the distribution $\mathcal{D}$.*

A.9 Koszul connection

Definition A.9.1 (Koszul connection). *Let $X, Y \in \Gamma(T\mathscr{M})$ be vector fields on $\mathscr{M}$, $f, g \in \Gamma(E)$ be sections of E and $k : \mathscr{M} \to \mathbb{R}$ a function on $\mathscr{M}$. The derivative operator $\nabla_X : \Gamma(E) \to \Gamma(E)$ is said to be a Koszul connection if it satisfies the following properties:*

1. $\nabla_X (f + g) = \nabla_X f + \nabla_X g$
2. $\nabla_{X+Y} f = \nabla_X f + \nabla_Y f$
3. $\nabla_{kX} f = k \nabla_X f$
4. $\nabla_X (kf) = X[k] f + k \nabla_X f$.

Remark: The action of a Koszul connection ∇ on any type of sections of E can be obtained from the action of ∇ on vector fields. For example, the action of ∇ on 1-forms can be written as:

$$(\nabla_X \alpha)(Y) = X[\alpha(Y)] - \alpha(\nabla_X Y) \tag{A.9.12}$$

where $X, Y \in \Gamma(T\mathscr{M})$ and $\alpha \in \Omega^1(\mathscr{M})$.

Definition A.9.2 (Koszul torsion). *Given a Koszul connection $\nabla_X : \Gamma(E) \to \Gamma(E)$, the Koszul torsion is the 2-form on $\mathscr{M}$ with values in $\Gamma(T\mathscr{M})$ defined as:*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{A.9.13}$$

where $X, Y \in \Gamma(T\mathscr{M})$.

Definition A.9.3 (Koszul curvature). *The curvature of a Koszul connection $\nabla_X : \Gamma(E) \to \Gamma(E)$ is the 2-form on $\mathscr{M}$ with values in* $\mathrm{End}(\Gamma(E))$ *satisfying*

$$R(X, Y; f) = \nabla_X \nabla_Y f - \nabla_Y \nabla_X f - \nabla_{[X,Y]} f \tag{A.9.14}$$

with $X, Y \in \Gamma(T\mathscr{M})$ and $f \in \Gamma(E)$.

Definition A.9.4 (Metric compatibility). *A Koszul connection ∇ is said compatible with the metric $\langle \cdot , \cdot \rangle : T\mathscr{M} \times T\mathscr{M} \to \mathbb{R}$ if $\forall X, Y, Z \in \Gamma(T\mathscr{M})$, we have:*

$$X[\langle Y, Z \rangle] = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle .$$

Proposition A.9.5 (Symmetry relations of the Koszul curvature). *Let ∇ be a Koszul connection and denote T and R its torsion and curvature, respectively. Let $\langle \cdot, \cdot \rangle : \Gamma\left(\vee^2 T^*\mathscr{M}\right) \to \Gamma(T\mathscr{M})$ be a covariant metric on $\mathscr{M}$. One assumes that ∇ is torsion-free and compatible* *with the metric $\langle \cdot, \cdot \rangle$. Then, the curvature tensor satisfies the following symmetry relations:*

$$R(X, Y; Z, W) = -R(Y, X; Z, W) \tag{A.9.15}$$

$$R(X, Y; Z) + R(Y, Z; X) + R(Z, X; Y) = 0 \tag{A.9.16}$$

$$R(X, Y; Z, W) = -R(X, Y; W, Z) \tag{A.9.17}$$

$$R(X, Y; Z, W) = R(Z, W; X, Y) \tag{A.9.18}$$

$\forall X, Y, Z, W \in \Gamma(T\mathscr{M})$, where $R(X, Y; Z, W) \equiv \langle R(X, Y; Z), W \rangle$.

Proposition A.9.6 (Levi-Civita connection)**.** *Let $\langle \cdot, \cdot \rangle : \Gamma\left(\vee^2 T^*\mathscr{M}\right) \to \Gamma(T\mathscr{M})$ be a maximal rank covariant metric on $\mathscr{M}$. There is a unique torsion-free Koszul connection ∇ compatible with $\langle \cdot, \cdot \rangle$, called the Levi-Civita connection.*

Definition A.9.7 (∇-preserving vector field)**.** *Let ∇ be a Koszul connection and $X \in \Gamma(T\mathscr{M})$ a vector field. X is called a ∇-preserving vector field if $\mathcal{L}_X \nabla = 0$ i.e. if $[X, \nabla_Y Z] = \nabla_{[X,Y]} Z + \nabla_Y [X, Z]$, $\forall Y, Z \in \Gamma(T\mathscr{M})$.*

Definition A.9.8 (Killing vector field)**.** *Let $\langle \cdot, \cdot \rangle : \Gamma\left(\vee^2 T^*\mathscr{M}\right) \to \Gamma(T\mathscr{M})$ be a covariant metric on $\mathscr{M}$ and $X \in \Gamma(T\mathscr{M})$ a vector field satisfying $\mathcal{L}_X \langle \cdot, \cdot \rangle = 0$ i.e. $X[\langle Y, Z \rangle] = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle$, $\forall Y, Z \in \Gamma(T\mathscr{M})$. Then X is called a Killing vector field of the metric $\langle \cdot, \cdot \rangle$.*

Proposition A.9.9. *Let $\langle \cdot, \cdot \rangle : \Gamma\left(\vee^2 T^*\mathscr{M}\right) \to \Gamma(T\mathscr{M})$ be a maximal rank covariant metric on $\mathscr{M}$ and denote ∇ its Levi-Civita connection. A vector field X is a Killing vector field for $\langle \cdot, \cdot \rangle$ if and only if $\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0$.*

Proposition A.9.10. *Let $\langle \cdot, \cdot \rangle : \Gamma\left(\vee^2 T^*\mathscr{M}\right) \to \Gamma(T\mathscr{M})$ be a maximal rank covariant metric on $\mathscr{M}$ and denote ∇ its Levi-Civita connection. A vector field X is a Killing vector field for $\langle \cdot, \cdot \rangle$ if and only if it is ∇-preserving.*

Definition A.9.11 (Geodesic vector field)**.** *Let ∇ be a Koszul connection and $X \in \Gamma(T\mathscr{M})$ a vector field. The vector field X is said geodesic if it satisfies $\nabla_X X = \kappa X$ with $\kappa \in C^\infty(\mathscr{M})$. It is said affine geodesic if $\nabla_X X = 0$.*

Definition A.9.12 (Recurrent tensor)**.** *A tensor $T \in \Gamma(E)$ is said recurrent for the Koszul connection ∇ if it satisfies the relation $\nabla_X T = \omega(X)\, T$ for some 1-form $\omega \in \Omega^1(\mathscr{M})$ called the recurrence 1-form.*

Proposition A.9.13. *Let $\langle \cdot, \cdot \rangle : \Gamma\left(\vee^2 T^*\mathscr{M}\right) \to \Gamma(T\mathscr{M})$ be a nondegenerate covariant metric on $\mathscr{M}$ and denote ∇ its associated Levi-Civita connection. Furthermore, let $\xi, X, Y \in \Gamma(T\mathscr{M})$ be three vector fields on $\mathscr{M}$ and denote $\psi \in \Omega^1(\mathscr{M})$ the 1-form dual to ξ: $\psi \equiv g(\xi)$. The following relation holds:*

$$d\psi(X, Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)\,. \tag{A.9.19}$$

Appendix B

Relativistic and Nonrelativistic Lie algebras

B.1 Kinematical algebras

In a seminal paper [75], Bacry and Levy-Leblond classified the various possible kinematical Lie algebras *i.e.* the Lie algebras encoding the infinitesimal kinematical symmetries of any free particle. Each of these symmetries takes its origin in a physical property of spacetime:

Physical requirement	**Generator**	**Transformation**	**Dimension**
Spacetime homogeneity	H	Time translation	1
	P_i	Spatial translations	d
Space isotropy	J_{ij}	Spatial rotations	$\frac{d(d-1)}{2}$
Relativity principle	K_i	Inertial boosts	d

Bacry and Levy-Leblond showed that under the following assumptions:

1. Space is isotropic *i.e.* infinitesimal generators transform under rotations as the
   (a) scalar representation for H:

$$[H, J] = 0$$

(b) vector representation for P, K:

$$[P, J] \sim P$$
$$[K, J] \sim K$$

(c) adjoint representation for J:

$$[J, J] \sim J$$

2. Space and time reversal operators Π and T are automorphisms of the kinematical Lie algebra

there exists a limited set of possible kinematical Lie algebras [1]:

1. (anti)-de Sitter
2. Poincaré
3. Galilei
4. Newton-Hooke
5. Carroll

Interestingly, the authors also showed that all of these Lie algebras can be derived from the (anti)-de Sitter algebra via a İnönü-Wigner contraction (*cf.* [156], See also [157] for an historical account).

B.2 Adjoint representations for relativistic and nonrelativistic groups

B.2.1 Poincaré group

Poincaré algebra :

$$\begin{aligned} [P_\mu, J_{\rho\sigma}] &= i\left(\eta_{\mu\sigma} P_\rho - \eta_{\mu\rho} P_\sigma\right) && \text{(B.2.1)} \\ [J_{\mu\nu}, J_{\rho\sigma}] &= i\left(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\sigma} J_{\mu\rho}\right) && \text{(B.2.2)} \end{aligned}$$

- $x \in \mathfrak{p}$; $x = x^i P_i$
- $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\left(h\right)\left(x\right) = \mathbf{R}^i_{\ j} x^j P_i$

1. We omit of the classification the “static” Lie algebra describing an infinite mass particle and for which all the commutators which do not imply J vanish as well as other Lie algebras isomorphic to the one stated below with $P \leftrightarrow K$.

- $a \in \mathfrak{p}^*$; $a = a_i P^{i*}$
- $\bar{\mathrm{A}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,(a) = a_j \mathbf{R}^{T j}{}_{i} P^{i*}$
- $g^{-1} \in \vee^2 \mathfrak{p}$; $g^{-1} = g^{ij} P_i \vee P_j$
- $\tilde{\mathrm{A}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\left(g^{-1}\right) = g^{ij} \mathbf{R}^k_i \mathbf{R}^l{}_j P_k \vee P_l$
- $g \in \vee^2 \mathfrak{p}^*$; $g = g_{ij} P^{i*} \vee P^{j*}$
- $\bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,(g) = g_{ij} \mathbf{R}^{T i}{}_{k} \mathbf{R}^{T j}{}_{l} P^{k*} \vee P^{l*}$

Examples:

- $\tilde{\mathrm{A}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,\delta^{ij} P_i \vee P_j = \delta^{ij} P_i \vee P_j$
- $\bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,\delta_{ij} P^{i*} \vee P^{j*} = \delta_{ij} P^{i*} \vee P^{j*}$

B.2.2 Weyl group

Weyl algebra :

$$[P_\mu, J_{\rho\sigma}] = i\left(\eta_{\mu\sigma} P_\rho - \eta_{\mu\rho} P_\sigma\right) \quad \text{(B.2.3)}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\sigma} J_{\mu\rho}\right) \quad \text{(B.2.4)}$$

$$[D, P_\mu] = -i P_\mu \quad \text{(B.2.5)}$$

- $x \in \mathfrak{p}$; $x = x^i P_i$
- $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,(x) = \lambda \mathbf{R}^i{}_j x^j P_i$
- $a \in \mathfrak{p}^*$; $a = a_i P^{i*}$
- $\bar{\mathrm{A}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,(a) = a_j \lambda^{-1} \mathbf{R}^{T j}{}_{i} P^{i*}$
- $g^{-1} \in \vee^2 \mathfrak{p}$; $g^{-1} = g^{ij} P_i \vee P_j$
- $\tilde{\mathrm{A}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\left(g^{-1}\right) = \lambda^2 g^{ij} \mathbf{R}^k_i \mathbf{R}^l{}_j P_k \vee P_l$
- $g \in \vee^2 \mathfrak{p}^*$; $g = g_{ij} P^{i*} \vee P^{j*}$
- $\bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,(g) = \lambda^{-2} g_{ij} \mathbf{R}^{T i}{}_{k} \mathbf{R}^{T j}{}_{l} P^{k*} \vee P^{l*}$

Examples:

- $\tilde{\mathrm{A}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,\delta^{ij} P_i \vee P_j = \lambda^2 \delta^{ij} P_i \vee P_j$
- $\bar{\tilde{\mathrm{A}}}\mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\,(h)\,\delta_{ij} P^{i*} \vee P^{j*} = \lambda^{-2} \delta_{ij} P^{i*} \vee P^{j*}$

B.2.3 Galilei group

Galilei algebra :

$$\begin{aligned} [H, K_i] &= iP_i && \text{(B.2.6)} \\ [P_i, J_{jk}] &= i\left(\delta_{ik}P_j - \delta_{ij}P_k\right) && \text{(B.2.7)} \\ [K_i, J_{jk}] &= i\left(\delta_{ik}K_j - \delta_{ij}K_k\right) && \text{(B.2.8)} \\ [J_{ij}, J_{kl}] &= i\left(\delta_{ik}J_{jl} - \delta_{jk}J_{il} - \delta_{il}J_{jk} + \delta_{jl}J_{ik}\right) && \text{(B.2.9)} \end{aligned}$$

- $x \in \mathfrak{p}$; $x = x^0 H + x^i P_i$
- $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(x) = \begin{pmatrix} \mathbf{R} & \mathbf{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^i \\ x^0 \end{pmatrix} = x^0 H + \left(\mathbf{R}^i_j x^j + \mathbf{b}^i x^0\right) P_i$
- $a \in \mathfrak{p}^*$; $a = a_0 H^* + a_i P^{i*}$
- $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)(a) = \begin{pmatrix} a^i & a^0 \end{pmatrix} \begin{pmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{b} \\ 0 & 1 \end{pmatrix} = \left(a_0 - a_i \mathbf{R}^{T i}{}_j \mathbf{b}^j\right) H^* + a_j \mathbf{R}^{T j}{}_i P^{i*}$
- $g^{-1} \in \vee^2 \mathfrak{p}$; $g^{-1} = \begin{pmatrix} g^{ij} & g^{0i} \\ g^{0j} & g^{00} \end{pmatrix}$
- $\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)\left(g^{-1}\right) = \begin{pmatrix} \mathbf{R} & \mathbf{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g^{ij} & g^{0i} \\ g^{0j} & g^{00} \end{pmatrix} \begin{pmatrix} \mathbf{R} & 0 \\ \mathbf{b} & 1 \end{pmatrix}$

$$= \begin{pmatrix} g^{ij}\mathbf{R}^l_j\mathbf{R}^k_i + g^{0i}\mathbf{R}^k_i\mathbf{b}^l + g^{0i}\mathbf{R}^l_i\mathbf{b}^k + g^{00}\mathbf{b}^k\mathbf{b}^l & g^{0i}\mathbf{R}^k_i + g^{00}\mathbf{b}^k \\ g^{0i}\mathbf{R}^l_i + g^{00}\mathbf{b}^l & g^{00} \end{pmatrix}$$

- $g \in \vee^2 \mathfrak{p}^*$; $g = \begin{pmatrix} g_{ij} & g_{0i} \\ g_{0j} & g_{00} \end{pmatrix}$
- $\tilde{\bar{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}(h)(g) = \begin{pmatrix} \mathbf{R}^T & 0 \\ -\mathbf{R}^T\mathbf{b} & 1 \end{pmatrix} \begin{pmatrix} g_{ij} & g_{0i} \\ g_{0j} & g_{00} \end{pmatrix} \begin{pmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{b} \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} g_{ij}\mathbf{R}^{T i}{}_k\mathbf{R}^{T j}{}_l & g_{0i}\mathbf{R}^{T i}{}_k - g_{ij}\mathbf{R}^{T i}{}_k\mathbf{R}^{T j}{}_l\mathbf{b}^l \\ g_{0i}\mathbf{R}^{T i}{}_l - g_{ij}\mathbf{R}^{T i}{}_l\mathbf{R}^{T j}{}_k\mathbf{b}^k & g_{00} - 2g_{0i}\mathbf{R}^{T i}{}_j\mathbf{b}^j + g_{ij}\mathbf{R}^{T i}{}_k\mathbf{b}^k\mathbf{R}^{T j}{}_l\mathbf{b}^l \end{pmatrix}$$

Examples:

- $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(H) = H + \mathbf{b}^i P_i$

- $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)(H^*) = H^*$
- $\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)\begin{pmatrix}\delta^{ij} & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\delta^{kl} & 0\\ 0 & 0\end{pmatrix}$
- $\bar{\tilde{\mathrm{Ad}}}_{\mathfrak{g}/\mathfrak{h}}(h)\begin{pmatrix}\delta_{ij} & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\delta_{kl} & -\delta_{kl}\mathbf{b}^l\\ -\delta_{kl}\mathbf{b}^k & \delta_{kl}\mathbf{b}^k\mathbf{b}^l\end{pmatrix}$

B.2.4 Bargmann group

Bargmann Algebra :

$$\begin{aligned}
[H, K_i] &= iP_i && \text{(B.2.10)}\\
[P_i, J_{jk}] &= i\left(\delta_{ik}P_j - \delta_{ij}P_k\right) && \text{(B.2.11)}\\
[P_i, K_j] &= i\delta_{ij}M && \text{(B.2.12)}\\
[K_i, J_{jk}] &= i\left(\delta_{ik}K_j - \delta_{ij}K_k\right) && \text{(B.2.13)}\\
[J_{ij}, J_{kl}] &= i\left(\delta_{ik}J_{jl} - \delta_{jk}J_{il} - \delta_{il}J_{jk} + \delta_{jl}J_{ik}\right) && \text{(B.2.14)}
\end{aligned}$$

- $x \in \mathfrak{p}$; $x = x^0H + x^iP_i + x^MM$
- $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(x) = \begin{pmatrix}\mathbf{R} & \mathbf{b} & 0\\ 0 & 1 & 0\\ -\mathbf{b}^T\mathbf{R} & -\frac{1}{2}\mathbf{b}^T\mathbf{b} & 1\end{pmatrix}\begin{pmatrix}x^i\\ x^0\\ x^M\end{pmatrix}$

$= x^0H + \left(\mathbf{R}^i_{\ j}x^j + \mathbf{b}^ix^0\right)P_i + \left(x^M - \mathbf{b}^T_k\mathbf{R}^k_{\ j}x^j - \frac{1}{2}\mathbf{b}^T_k\mathbf{b}^kx^0\right)M$

- $a \in \mathfrak{p}^*$; $a = a_0H^* + a_iP^{i*} + a_MM^*$
- $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)(a) = (a_i, a_0, a_M)\begin{pmatrix}\mathbf{R}^T & -\mathbf{R}^T\mathbf{b} & 0\\ 0 & 1 & 0\\ \mathbf{b}^T & -\frac{1}{2}\mathbf{b}^T\mathbf{b} & 1\end{pmatrix}$

$= \left(a_0 - a_i{\mathbf{R}^T}^i_{\ j}\mathbf{b}^j - \frac{1}{2}\mathbf{b}^T\mathbf{b}a_M\right)H^* + \left(a_j{\mathbf{R}^T}^j_{\ i} + a_M\mathbf{b}^T_i\right)P^{i*} + a_MM^*$

Examples:

- $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(H) = H + \mathbf{b}^iP_i - \frac{1}{2}\mathbf{b}^T\mathbf{b}M$
- $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(M) = M$
- $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)(H^*) = H^*$
- $\bar{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}(h)(M^*) = M^* - \mathbf{b}^T_k\mathbf{R}^k_{\ j}P^{j*} - \frac{1}{2}\mathbf{b}^T\mathbf{b}H^*$

- $\tilde{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{h}}\,(h)\begin{pmatrix}\delta_{ij} & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}\delta_{kl} & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$

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Notations and Conventions

- $T_x\mathscr{M}$: Tangent space at the point $x \in \mathscr{M}$
- $T_x^*\mathscr{M}$: Dual tangent space at the point $x \in \mathscr{M}$
- $T\mathscr{M}$: Tangent bundle of $\mathscr{M}$
- $T^*\mathscr{M}$: Dual tangent bundle of $\mathscr{M}$
- $\Gamma(T\mathscr{M})$: Space of vector fields on $\mathscr{M}$
- $\Omega^p(\mathscr{M})$: Space of p-forms on $\mathscr{M}$
- $\Gamma\left(\vee^2 T\mathscr{M}\right)$: Space of fields of contravariant symmetric bilinear forms on $\mathscr{M}$
- $\Gamma\left(\vee^2 T^*\mathscr{M}\right)$: Space of fields of covariant symmetric bilinear forms on $\mathscr{M}$
- $X_x \in T_x\mathscr{M}$: Vector field $X \in \Gamma(T\mathscr{M})$ evaluated at the point $x \in \mathscr{M}$
- Ker ψ: Subbundle of $T\mathscr{M}$ spanned by vector fields in $\Gamma(T\mathscr{M})$ annihiliated by the 1-form $\psi \in \Omega^1(\mathscr{M})$
- Ann N: Subbundle of $T^*\mathscr{M}$ spanned by 1-forms in $\Omega^1(\mathscr{M})$ annihilating the vector field $N \in \Gamma(T\mathscr{M})$
- $FO(\mathscr{M})$: Space of fields of observers on $\mathscr{M}$
- $\mathrm{End}(V)$: Space of linear maps on the vector space V
- $\bigotimes V \otimes \bigotimes V^*$: Space of multilinear maps on the vector space V
- $g(X) \equiv g(X, \cdot) \in \Omega^1(\mathscr{M})$, with $g \in \Gamma\left(\vee^2 T^*\mathscr{M}\right)$ and $X \in \Gamma(T\mathscr{M})$
- $\Phi_{(\mu\nu)} \equiv \frac{1}{2}(\Phi_{\mu\nu} + \Phi_{\nu\mu})$: Symmetrisation with weight one
- $\Phi_{[\mu\nu]} \equiv \frac{1}{2}(\Phi_{\mu\nu} - \Phi_{\nu\mu})$: Anti-Symmetrisation with weight one

Glossary of structures

Leibnizian structure $\mathscr{L}(\mathscr{M}, \psi, h)$ *cf.* Definition 3.2.1 (resp. 3.2.2)

- Spacetime manifold $\mathscr{M}$ endowed with :
- – Absolute clock $\psi \in \Omega^1(\mathscr{M})$
- – Absolute rulers $h \in \Gamma\left(\vee^2 T\mathscr{M}\right)$ with Rad $h =$ Span ψ (resp. $\gamma \in \Gamma\left(\vee^2 \operatorname{Ker}\psi\right)$)

Aristotelian structure $\mathscr{A}(\mathscr{M}, \psi, h)$ *cf.* Definition 3.2.15

- Leibnizian structure with $\psi \wedge d\psi = 0$

Augustinian structure $\mathscr{S}(\mathscr{M}, \psi, h)$ *cf.* Definition 3.2.16

- Leibnizian structure with $d\psi = 0$

Lagrangian structure $\mathcal{L}(\mathscr{M}, \psi, [g])$ *cf.* Definition 3.2.40

- Leibnizian structure endowed with a Lagrangian class of metrics $[g]$

Galilean manifold $\mathscr{G}(\mathscr{M}, \psi, h, \nabla)$ *cf.* Definition 3.2.18 (resp. 3.2.19)

- Leibnizian structure supplemented with a Koszul connection ∇ satisfying the compatibility conditions
- – $\nabla\psi = 0$
- – $\nabla h = 0$ (resp. $\nabla\gamma = 0$)

Newtonian manifold $\mathscr{N}(\mathscr{M}, \psi, h, \nabla)$ *cf.* Definition 3.2.33

- Augustinian structure supplemented with a compatible torsionfree Koszul connection ∇ satisfying the Duval-Künzle condition (*cf.* Definition 3.2.28)

Horizontal manifold $\mathscr{H}(\mathscr{M}, \psi, h, \nabla)$ *cf.* Definition 3.2.46

- Aristotelian structure supplemented with a torsionfree Koszul connection on $\mathscr{M}$ satisfying Axioms 1-3 of Proposition 3.2.45

Platonic manifold $\mathscr{P}(\mathscr{M}, \psi, h, \nabla)$ *cf.* Definition 3.2.49

- Aristotelian structure supplemented with a torsionfree Koszul connection ∇ whose coefficients are given by (3.2.63)

Résumé :

Bien qu'ayant vu le jour dans un cadre dit relativiste avec l'avènement de la théorie de la relativité générale, le lien intime existant entre géométrie de l'espace-temps d'une part, et gravitation d'autre part, peut se voir étendu aux théories dites nonrelativistes, l'exemple paradigmatique en étant la reformulation géométrique de la gravitation Newtonienne initiée par E. Cartan. De tels espace-temps nonrelativistes diffèrent structurellement de leurs homologues relativistes, ces disparités étant le plus naturellement expliquées en réinterprétant ces premiers comme réduction dimensionnelle d'espace-temps relativistes privilégiés.

L'ambition de cette thèse est double :

Dans une première partie, nous nous intéressons à une généralisation de la classe d'espace-temps relativistes permettant le formalisme ambiant, étudions leur interprétation géométrique ainsi que la classe élargie de structures nonrelativistes pouvant y être plongées.

La seconde partie de ce manuscrit concerne le point de vue, informé par la théorie des groupes, que porte E. Cartan sur la géométrie différentielle et plus précisément l'éclairage que projettent les géométries de Cartan sur les structures nonrelativistes, à la fois dans leur définition intrinsèque et dans leur relation avec des structures relativistes au travers du formalisme ambiant.

Mots clés :

Symétries Nonrelativistes, Eisenhart Lift, Gravitation de Newton-Cartan, Réduction Dimensionnelle, Formalisme Ambiant, Géométrie de Cartan.

Abstract :

With the advent of general relativity, the profound interaction between the geometry of spacetime and gravitational phenomena became a truism of modern physics. However, the intimate relationship between spacetime geometry and gravitation is by no means restricted to relativistic physics but can in fact be successfully applied to nonrelativistic physics, the paradigmatic example being E. Cartan geometrisation of Newtonian gravity. This geometrisation of nonrelativistic gravitation involves some nonrelativistic structures whose discrepancies in comparison with their relativistic peers are better understood when embedded inside specific classes of relativistic gravitational waves.

The ambition of this Doctoral Thesis is twofold:

In a first part, we discuss a generalisation of the class of gravitational waves allowing the embedding of nonrelativistic features, explore their geometric properties and the new nonrelativistic structures emerging from this study.

In a second part, we advocate how the group-theoretically oriented approach of Cartan to differential geometry can shed new light on nonrelativistic structures, both in an intrinsic and ambient fashion.

Keywords :

Nonrelativistic Symmetries, Eisenhart Lift, Newton-Cartan Gravity, Dimensional Reduction, Ambient Formalism, Cartan Geometry.